國立交通大學 101 學年度碩士班考試入學試題

科目：微分方程與線性代數(1212)　　考試日期：101 年 2 月 17 日 第 2 節

系所班別：電機工程學系　　組別：電機系　　

【可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. (15%) Please determine whether the following statements are TRUE or FALSE. No explanations are needed.

   (a) (3%) Suppose $A\mathbf{x} = \mathbf{b}$ is a consistent system and $\mathbf{v}_1$ and $\mathbf{v}_2$ are two solutions, where A is an $m \times n$ matrix. Then $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ must be a solution for the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

   (b) (3%) Let A and B be $m \times n$ matrices. If $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x} \in \mathcal{R}^n$, then A and B are equal.

   (c) (3%) Let A be an $n \times n$ matrix. If A is invertible, then any subset of $\mathcal{R}^n$ formed by the columns vectors of A is linearly independent.

   (d) (3%) Let $\mathcal{S} = \{\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_k\}$ be a subset of $\mathcal{R}^n$ and $T : \mathcal{R}^n \to \mathcal{R}^m$ be a linear transformation. When $\{T(\mathbf{a}_1), T(\mathbf{a}_2), \cdots, T(\mathbf{a}_k)\}$ is linear independent, $\mathcal{S}$ is not necessarily linearly independent.

   (e) (3%) Let A be an $m \times n$ matrix. If the number of pivot numbers of A is less than m, then $A\mathbf{x} = \mathbf{b}$ has more than one solution.

2. (10%) Let $\mathcal{W}_1, \mathcal{W}_2, \cdots, \mathcal{W}_m$ be subspaces of $\mathcal{R}^n$ such that $\mathcal{W}_{i+1}$ is a subset of $\bigcap_{j=1}^{i} \mathcal{W}_j^{\perp}$, where $\mathcal{W}_j^{\perp}$ is the orthogonal complement of $\mathcal{W}_j$, for $i = 1, 2, \cdots, m-1$. Suppose the orthogonal projection matrix for $\mathcal{W}_i$ is P_i. Determine the orthogonal projection matrix for $V = \{\mathbf{v} \in \mathcal{R}^n : \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_m,\ \text{where } \mathbf{v}_i \in \mathcal{W}_i\}$.

3. (a) (8%) Find the eignevalues and eigenvectors for

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

   (b) (3%) How can you tell whether a matrix is invertible from its eigenvalues?

   (c) (3%) How can you tell whether two matrices are similar from their eigenvalues?

4. (a) (8%) For the following matrix, find the bases for its row space and nullspace.

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 4 & 5 & 6 & 3 \\ 3 & 4 & 4 & 1 \end{bmatrix}$$

   (b) (3%) In $\mathcal{R}^3$, is xy plane orthogonal to xz plane ? Explain it.

5. (10%) Find the solutions $y(x)$ to the following initial value problems

   (a) (5%)

$$x^2 y'(x) - 2xy(x) = 3y^3(x), \quad y(1) = \frac{1}{2}.$$

(b) (5%)

$$xy'(x) - y(x) = x^2 e^x, \quad y(0) = 0.$$

6. (10%) Consider the following second order ordinary differential equation:

$$xy''(x) - (x+2)y'(x) + 2y(x) = 0.$$

Determine the solution $y(x)$ satisfying the conditions $y''(0) = 1$ and $y'''(0) = 0$.

7. (15%) For the following initial value problem

$$y''(t) + y(t) = f(t) = \sum_{n=0}^{\infty} u(t-n), \quad y(0) = 1,\ y'(0) = 0, \tag{1}$$

where $u(t)$ is the unit-step function,

(a) (5%) Find the Laplace transform of $f(t)$ and determine its region of convergence.

(b) (10%) Solve the above ordinary differential equation (1) for $y(t)$.

8. (15%) Let $\mathbf{x}(t)$ be a length-3 vector of functions in t that satisfies the following system of linear differential equations:

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} e^{4t}, \quad \mathbf{x}(1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Find the solution $\mathbf{x}(t)$.