考試科目	計算機數學	系所別	資訊科學系	考試時間	2 月 9 日(三) 第 4 節

I. 離散數學：60% (第 1~9 題)

II. 線性代數：40% (第 10~12 題)

請書寫必要的解題過程，僅提供答案而無必要過程，將無法獲得該題滿分。可使用中文或英文作答，力求書寫工整，如字跡潦草，無法閱讀，將影響評分。

1. (4%) Which of the following pairs of values (n, k) (if any) make the following statement true. If 50 students are distributed into n classrooms, there must be a classroom with at least k students?
   (a) (3, 15), (b) (70, 3), (c) (15, 3), (d) (7, 8), (e) (51, 2)
2. (5%) Which of the following asymptotic notation(s) is/are correct for the equation $3N^3\log(N) + 4N\log(N)$?
   (a) $\Theta(N^2 log(N))$, (b) $O(N^3)$, (c) $\Omega(N^2\log(N))$, and (d) $O(N^3\log(N^2))$.
3. (5%) If A and B are disjoint events, and $P(A) = 0.4$ and $P(B) = 0.2$, what is $P(B|A)$?.
4. (6%) Please use a recursion tree to determine a good asymptotic upper bound on the recurrence
$$T(n) = 3T(\left\lfloor\frac{n}{2}\right\rfloor) + n$$
5. (4%) Please prove that $n(n+1)(n+2)$ is divisible by 3 using mathematical induction.
6. (10%) Suppose that the two continuous random variables X and Z are statistically independent. Please prove that the mean and variance of their sum satisfy (please use definition of the expected value and variance):
   (a) (5%) E[X + Z] = E[X] + E[Z].
   (b) (5%) Var[X + Z] = Var[X] + Var[Z].
7. (6%) Please prove $222|(2^a - 2^b)$ for some positive integers, a and b.
8. (10%) Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, for $n \geq 2$, $a_0 = 1, a_1 = 0$.
9. (10%) Find the set of all solutions x to the system of two congruences below:
$$7x \equiv 14 (\text{mod } 6) \text{ and } 5x \equiv 3 (\text{mod } 6).$$

10. (5%) True or false.
    (a) If a rectangle matrix A is invertible, A could have a zero singular value.
    (b) If a matrix B is similar to A, then B has the same eigenvectors as A.
    (c) Any symmetric matrix is similar to a diagonal matrix.
    (d) Any matrix which is similar to a diagonal matrix is symmetric.
    (e) Every positive definite matrix is nonsingular.

11. (15%) Suppose a rectangle matrix A has full rank.
(a) (4%) Please derive the best square solution $\tilde{x}$ to $Ax = b$ step by step.
(b) (6%) Which fundamental subspace associated with A is the projection vector p in, where $p = A\tilde{x}$? Which

考 試 科 目	計算機數學	系 所 別	資訊科學系	考 試 時 間	2 月 9 日(三) 第 4 節

fundamental subspace is $p-b$ in then?

(c) (5%) Find the projection matrix P onto the column space of $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}$.

12. (20%) Given a recurrence relation: $(M^TM + 4I)x_{n+1} = (M^TM - 4I)x_n$, where $M \in R^{5\times 3}$ (a real 5×3 matrix) and n is a positive integer.

(a) (5%)Let $x_n = A^n x_0$. Please find out A (independent of any x_i, $i \in N \cup \{0\}$)

(b) (5%) Let λ be an eigenvalue of M^TM, corresponding to the eigenvector λ. Please find an eigen pair of A. (an eigen pair means an eigenvector and its associated eigenvalue)

(c) (5%) Given all the information above, choose all the possible behaviors of x_n for a large n from (1) decaying to zero, (2) oscillating but not growing or decaying in length, (3) going to a nonzero constant vector, and (4) going to a nonzero constant vector. Explain your answers by discussing $A's$ eigenvalue.

(d) (5%) If $x_1 = 0$ for a nonzero x_0, please find one singular value of M.

備 註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。