※ 注意：請於試卷內之「非選擇題作答區」標明題號依序作答。

**Instructions:**

- This exam has two parts.
  Part I consists of partial-credit problems. Steps must be shown. Answers must be justified.
  Part II consists of fill-in-the-blank problems. Only the clearly labeled answers will be graded.
- No electronic devices or computer algebra systems allowed for this exam.
- Usage of any theorem/formula must be clearly stated.

**Part I**: 15 points for each problem. Make sure your work is organized and logical.

1. Sketch the curve $y = \dfrac{(x+1)^2}{\sqrt{x^2-1}}$ for $x > 1$ and find its asymptotes (horizontal, vertical, slant). Find the intervals of increase/decrease and concavity.

2. Evaluate the improper integral $\displaystyle\int_6^\infty \frac{81}{x^3(x^2-9)}\,dx$ with the specified methods.

   (a) Use trigonometric substitution $x = 3\sec\theta$ first. Then evaluate the trigonometric integral.

   (b) Find the partial fractions form of $\dfrac{81}{x^3(x^2-9)}$. Then evaluate the integral.

3. Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x,y,z,w) = x^2 + 4y + 6z + 8w$ subject to the constraints $x^2+y^2+z^2 = 5$ and $z + w = 2$.

4. Use Green's Theorem to calculate the vector line integral $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$, where
$$\mathbf{F}(x,y) = \frac{2xy\,\mathbf{i} + (y^2-x^2)\,\mathbf{j}}{(x^2+y^2)^2}$$
and $C$ is ANY positively oriented simple closed curve on $\mathbb{R}^2$ that encloses the origin.

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**Part II**: 4 points for each blank. Make sure you use the labels for the blanks.

5. Evaluate the limits: $\displaystyle\lim_{x\to\infty}\left[\sqrt{1+\frac{3}{x}}\right]^{\sqrt{x^2+1}} =$ __(1)__ , $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{n}{4n^2+k^2} =$ __(2)__ .

6. The gradient vector of the function $f(x,y,z) = (2x+3z)^y$ at the point $(1,1,1)$ is __(3)__ .
   Now suppose a particle at $(1,1,1)$ is moving on the surface $f(x,y,z) = 5$ with $\dfrac{dx}{dt} = 1$ and $\dfrac{dy}{dt} = -1$.
   Then $\dfrac{dz}{dt} =$ __(4)__ .

7. Evaluate the multiple integrals.
$$\int_0^2\int_{2y}^4 e^{-x^2}\,dx\,dy = \text{\_\_(5)\_\_}, \qquad \int_{-2}^2\int_0^{\sqrt{4-x^2}}\int_0^{\frac{1}{2}\sqrt{4-x^2-y^2}}(x^2+y^2+4z^2)\,dz\,dy\,dx = \text{\_\_(6)\_\_}.$$

8. Consider $f(x) = \displaystyle\int_x^{2x}\frac{\sin(t^3)}{t}\,dt$.
   Express $f(1)$ as an infinite series: $f(1) =$ __(7)__ .
   Define $f(0) = 0$. Then we can find the 2025-th derivative of $f$ at $x = 0$: $f^{(2025)}(0) =$ __(8)__ .

9. Find the interval(s) of $p$ where the expression converges.
   $\displaystyle\int_0^\infty \frac{x^p}{e^x-1}\,dx$ converges when $p$ is in __(9)__ . $\displaystyle\sum_{n=2}^\infty \frac{(2p+1)^n}{\sqrt{n}\ln(n+1)}$ converges when $p$ is in __(10)__ .

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