科目：工程數學 C(3005)　校系所組：中央大學電機工程學系(電子組、系統與生醫組)
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參考用

- 請將答案依下圖所示由上而下依序寫在答案卷的作答區的第一頁。
- 只要填寫考題所要求的答案，請勿附加計算過程。

從此處開始寫起	
一、(一)···	(二)···
二、(一)···	(二)···
三、(一)···	(二)···
四、(一)···	(二)···
五、(一)···	(二)···
⋮	

一、 (9 %) Let $a > b > 0$, and define $\mathbf{M} = a\mathbf{u}\mathbf{u}^{\mathrm{T}} + b\mathbf{I}_4$, where $\mathbf{u} = \begin{bmatrix}1 & 2 & 2 & 1\end{bmatrix}^{\mathrm{T}}$ and $\mathbf{I}_4$ is the identity matrix of dimension 4×4.

(一) (5 %) Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ be the eigenvalues of $\mathbf{M}$. Compute $\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4$.

(二) (4 %) Find the maximal singular value of matrix $\mathbf{M}$.

二、 (8 %) Assume that $(U, \langle\cdot,\cdot\rangle_U)$ and $(V, \langle\cdot,\cdot\rangle_V)$ are two Euclidean inner product spaces spanned by the ordered orthonormal bases $B_U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B_V = \{\mathbf{v}_1, \mathbf{v}_2\}$, respectively. Let $T : U \to V$ be a linear transformation from U into V, and the matrix representation of T with respect to B_U and B_V is given by

$$\mathbf{A} = \begin{bmatrix}1 & 0 & 1\\ 1 & 2 & -1\end{bmatrix}.$$

(一) (3 %) Let $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ and $\mathbf{v} = T(\mathbf{u})$. Compute the norm $\|\mathbf{v}\| = \sqrt{\langle\mathbf{v},\mathbf{v}\rangle_V}$.

(二) (5 %) Find the $\mathbf{u}_0 \in U$ such that (i) $T(\mathbf{u}_0) = \mathbf{v}_1 + 3\mathbf{v}_2$ and (ii) $\|\mathbf{u}_0\| = \sqrt{\langle\mathbf{u}_0,\mathbf{u}_0\rangle_U}$ is minimized.

三、 (一) (4%) Compute the projection matrix $\mathbf{P}$ and the projection vector $\mathbf{p}$ of $\mathbf{b}$ onto the column space of $\mathbf{A}$, where

$$\mathbf{A} = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 0 & 1\end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix}3\\ 5\\ 7\end{bmatrix}.$$

注意：背面有試題

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(二) (4%) Suppose that $\hat{\mathbf{x}} = \mathbf{y}$ minimizes the Euclidean norm $\|\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}\|$ among all possible $\hat{\mathbf{x}}$. Let $\mathbf{p} = \mathbf{A}\mathbf{y}$. Which of the four spaces (i.e., column space $C(\mathbf{A})$, row space $R(\mathbf{A})$, right null space $N(\mathbf{A})$ and left null space $N(\mathbf{A}^\top)$ of $\mathbf{A}$) always contains $\mathbf{p}$? Justify your answer.

四、(一) (4%) Find the determinant of $(\mathbf{AB})^{-2}$:

$$\mathbf{A} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}, \quad \mathbf{B} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ \tan(\theta) & 1 & 0 \\ \sin(\theta) & \cos(\theta) & 1 \end{bmatrix} \begin{bmatrix} \theta & \theta^2 & \theta^3 \\ 0 & 2 & 2\theta \\ 0 & 0 & 4\theta^{-1} \end{bmatrix}.$$

(二) (4%) Give the property of block determinant:

$$\det\left(\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}\right) = \det(\mathbf{A}) \cdot \det(\mathbf{D})$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ are some square matrices.

Use the above property to compute the determinant of the matrix:

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 4 \\ 9 & -3 & 25 & 15 \\ 1 & -5 & 12 & -16 \end{bmatrix}.$$

$$\left(\text{Hint: Change it to the form } \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & m & n \\ 0 & 0 & p & q \end{bmatrix}\right).$$

五、(一) (4%) Given a 4×4 matrix $\mathbf{P} = \mathbf{P}_{21}\mathbf{P}_{31}\mathbf{P}_{41}\mathbf{P}_{32}$, where $\mathbf{P}_{ij}$ is the basic 4×4 permutation matrix which exchanges row i with row j, find $\mathbf{P}^{-1}$.

(二) (4%) If a 3×4 matrix $\mathbf{A}$ has the vector $[4, 8, 1, 0]^\top$ as the only special solution to $\mathbf{Ax} = \mathbf{0}$, find the reduced row echelon matrix $\mathbf{R}$ of $\mathbf{A}$.

六、(一) (4%) Consider an arbitrary $m \times n$ matrix $\mathbf{A}$ with rank r, $r < m$ and $r < n$, and an arbitrary column vector $\mathbf{b}$ of m elements. Find the number of all possible solutions to the general linear equation $\mathbf{Ax} = \mathbf{b}$.

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（二） (5%) Consider an arbitrary $m \times n$ matrix $\mathbf{A}$ with rank r and an arbitrary $n \times k$ matrix B with rank s, find the tightest lower bound for the rank of matrix $\mathbf{AB}$.

七、 (8%) Let $\delta(t)$ be the delta function. Solve the following equations and plot $x(t)$.

（一） (4%) $x''(t) + 4x(t) = 4\sum_{n=0}^{\infty} \delta(t - n\pi)$, $x(0) = x'(0) = 0$.

（二） (4%) $\int_0^t x(t-u)\cos(u)\,du \;=\; t\sin(t)$.

八、 (10%) Consider the following linear system

$$\mathbf{x}'(t) \;=\; \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}.$$

（一） (3%) Find the complementary solution of the associated homogeneous system, i.e., if $f_1(t) = f_2(t) = 0$

（二） (4%) If $f_1(t) = f_2(t) = t^{-1}$, find the particular solution.

（三） (3%) Given the initial condition $\mathbf{x}(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in 八、（二）, find the solution $\mathbf{x}(t)$.

九、 (6%) Consider the following initial value problem

$$y''(x) - 2xy'(x) + 2y'(x) + 8y(x) \;=\; 0, \quad y(1) = 3,\ y'(1) = 0$$

Assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$ has a power series solution at $x = 0$, determine the values of $c_0, c_1, c_2, c_3, c_4, c_5$.

十、 (10%) Consider the following boundary value problem for $u(x,t)$ with $0 < x < \pi$ and $t > 0$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0. \qquad (1)$$

（一） (5%) Assuming $f(x) = x$, determine its Fourier sine series on interval $[0,\pi)$.

（二） (5%) Solve the boundary value problem (1) for $u(x,t)$ when $f(x) = x$.

十一、 (16%) Consider the following initial value problem

$$y''(x) + ay'(x) + by(x) = f(x), \quad y(0) = c,\ y'(0) = d,$$

where a, b, c, and d are constants. It is known that $y(x) = \sin(2x)$ when $f(x) = -3\sin(2x)$.

（一） (4%) Find the values of a, b, c, and d.

（二） (12%) Calculate the corresponding solution $y(x)$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ when $f(x) = \tan(x)$.