國立中山大學114學年度
碩士班考試入學招生考試試題

科目名稱：工程數學甲【電機系碩士班戊組選考、庚組、通訊所碩士班乙組選考、電波聯合碩士班選考】

—作答注意事項—

考試時間：100分鐘

- **考試開始鈴響前不得翻閱試題，並不得書寫、劃記、作答。**請先檢查答案卷（卡）之應考證號碼、桌角號碼、應試科目是否正確，如有不同立即請監試人員處理。
- **答案卷限用藍、黑色筆(含鉛筆)書寫、繪圖或標示**，可攜帶橡皮擦、無色透明無文字墊板、尺規、修正液（帶）、手錶(未附計算器者)。每人每節限使用一份答案卷，請衡酌作答。
- 答案卡請以2B鉛筆劃記，不可使用修正液（帶）塗改，未使用2B鉛筆、劃記太輕或污損致光學閱讀機無法辨識答案者，後果由考生自負。
- 答案卷（卡）應保持清潔完整，不得折疊、破壞或塗改應考證號碼及條碼，亦不得書寫考生姓名、應考證號碼或與答案無關之任何文字或符號。
- 可否使用計算機請依試題資訊內標註為準，如「可以」使用，廠牌、功能不拘，唯不得攜帶書籍、紙張（應考證不得做計算紙書寫）、具有通訊、記憶、傳輸或收發等功能之相關電子產品或其他有礙試場安寧、考試公平之各類器材入場。
- 試題及答案卷（卡）請務必繳回，未繳回者該科成績以零分計算。
- 試題採雙面列印，考生應注意試題頁數確實作答。
- 違規者依本校招生考試試場規則及違規處理辦法處理。

國立中山大學 114 學年度碩士班考試入學招生考試試題

科目名稱：工程數學甲【電機系碩士班戊組選考、庚組、通訊所碩士班乙組選考、電波聯合碩士班選考】題號：431002

※本科目依簡章規定「可以」使用計算機（廠牌、功能不拘）（混合題）　　

第 1-6 題為複選題，每題 5 分，總分 30 分。每題有 5 個選項，其中至少有 1 個是正確答案，答錯 1 個選項者，得 3 分；答錯 2 個選項者，得 1 分；答錯多於 2 個選項或未作答者，該題以零分計算。

1. Consider the linear system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] \in \mathbb{R}^{4\times3}$, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are column vectors of $\mathbf{A}$, and $\mathbf{b} \in \mathbb{R}^4$ is a non-zero vector. Suppose

$$\mathbf{a}_1 + 2\mathbf{a}_2 = 3\mathbf{a}_3,\ \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{b},\ \mathbf{a}_2 + 2\mathbf{a}_3 \neq \mathbf{0}.$$

Which of the following statements are true?
(A) The linear system has a finite number of solutions.
(B) $\text{rank}([\mathbf{A}, \mathbf{b}])$ is equal to $\text{rank}(\mathbf{A})$.
(C) The rank of $\mathbf{A}$ is less than or equal to 2.
(D) $\mathbf{x} = [1, 2, 3]^T$ is a solution of the linear system.
(E) The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are linearly dependent.

2. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Which of the following statements are true?
(A) $\text{rank}(\mathbf{A}) = 2$.
(B) $\text{trace}(\mathbf{A}) = 7$.
(C) $\det(\mathbf{A}) \neq 0$.
(D) 3 is an eigenvalue of $\mathbf{A}$.
(E) All eigenvalues of $\mathbf{A}$ are non-negative.

3. Let

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] = \mathbf{QR} = \mathbf{Q}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix},$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are column vectors of $\mathbf{A} \in \mathbb{R}^{3\times3}$, and $\mathbf{Q} \in \mathbb{R}^{3\times3}$ is an orthogonal matrix, i.e., $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T$ is the identity matrix. Which of the following statements are true?

(A) $\mathbf{a}_3^T\mathbf{a}_1 = 3$, (B) $\mathbf{a}_3^T\mathbf{a}_2 = 24$, (C) $\mathbf{a}_3^T\mathbf{a}_3 = 70$, (D) $\mathbf{a}_2^T\mathbf{a}_1 = 3$, (E) $\mathbf{a}_2^T\mathbf{a}_2 = 30$

4. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3],$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are column vectors of $\mathbf{A}$. Let $\mathbf{q}_3$ be the orthogonal projection of $\mathbf{a}_3$ onto $\text{span}(\mathbf{a}_1, \mathbf{a}_2)$, and $\mathbf{q}_3 = [q_{31}, q_{32}, q_{33}, q_{34}]^T$. Which of the following statements are true?

(A) $q_{31} = 4$, (B) $q_{32} = 4/3$, (C) $q_{33} = 10/3$, (D) $q_{34} = 22/3$, (E) $\mathbf{a}_1^T\mathbf{q}_3 = 20$.

5. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{A}\mathbf{A}^T = \mathbf{I}$, and $\mathbf{A}$ is a non-zero matrix. Which of the following statements are true?

(A) $\text{trace}(\mathbf{A}) \neq 0$.
(B) $\det(\mathbf{A}) = 1$.
(C) $\text{rank}(\mathbf{A}) = n$.
(D) If $n = 3$, then $\text{trace}(\mathbf{A}) \leq 3$.
(E) All eigenvalues of $\mathbf{A}$ are real numbers.

6. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, where $R(\mathbf{A})$ denotes the column space of $\mathbf{A}$, $N(\mathbf{A})$ denotes the null space of $\mathbf{A}$, and $\dim(S)$ denotes the dimension of a subspace S. Which of the following statements are true?

(A) If $\mathbf{A}$ has linearly independent columns, then $\mathbf{A}\mathbf{A}^T$ is nonsingular.
(B) $\text{rank}(\mathbf{A}^T) + \dim(N(\mathbf{A}^T)) = m$.
(C) $R(\mathbf{A}\mathbf{A}^T) = R(\mathbf{A}^T\mathbf{A})$.
(D) If $\mathbf{A}\mathbf{A}^T$ is nonsingular, then $\mathbf{A}^T\mathbf{A}$ is also nonsingular.
(E) It is possible for a matrix $\mathbf{A}$ to have $[3, 4, 5]^T$ in $R(\mathbf{A})$ and $[1, -1, 2]^T$ in $N(\mathbf{A}^T)$.

第7-13題為單選題，總分35分。每題答對5分，答錯或未作答者以0分計。

7. Find the inverse Laplace transform of $\frac{-5}{s+16}$.

(A) $5e^{16t}$　(B) $-5e^{16t}$　(C) $-5e^{-16t}$　(D) $5e^{-16t}$

8. Which of the following is **wrong**?

(A) The Fourier transform of a rectangular pulse of width T and height 1 is $T\text{sinc}\left(\frac{\omega T}{2}\right)$.
(B) The Fourier transform of $u(t)$ (unit step function) is $\pi\delta(\omega)$.
(C) If $f(x)$ is odd, its Fourier series contains only sine terms.
(D) The initial value theorem for the Laplace transform is $\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$.

9. Which of the following statements about the Gibbs phenomenon is **wrong**?

(A) It occurs near discontinuities when approximating a function using its Fourier series.
(B) The oscillatory behavior in the Gibbs phenomenon disappears completely as more terms are added to the Fourier series.
(C) The Gibbs phenomenon results in a characteristic overshoot near discontinuities, even with a large number of terms.
(D) The oscillations caused by the Gibbs phenomenon become narrower as more terms are included.

10. Suppose we are given the following information about a signal $x[n]$:

a. $x[n]$ is a real and even signal.
b. $x[n]$ has period N=10 and Fourier coefficients a_k.
c. $a_{11} = 5$.
d. $\frac{1}{10}\sum_{n=0}^{9}|x[n]|^2 = 50$.

Show that $x[n] = A\cos(Bn + C)$, and specify numerical values for the constants A, B, and C.

(A) $A = 10, B = \frac{\pi}{5}, C = 0$
(B) $A = 5, B = \frac{\pi}{5}, C = \frac{\pi}{2}$
(C) $A = 10, B = \frac{\pi}{5}, C = \frac{\pi}{2}$
(D) $A = 5, B = \frac{\pi}{5}, C = 0$

11. What is the Fourier transform of $\delta(t+1) + \delta(t-1)$?
(A) $e^{-j\omega} + e^{j\omega}$ (B) $e^{-j\omega} - e^{j\omega}$ (C) $2jcos(\omega)$ (D) $2jsin(\omega)$

12. The Fourier transform $X(e^{j\omega})$ of the discrete-time signal $x[n] = \cos\left(\frac{3\pi}{5}n\right)$ is:
(A) $\frac{\pi}{2}\delta\left(\omega - \frac{3\pi}{5}\right) + \frac{\pi}{2}\delta\left(\omega + \frac{3\pi}{5}\right)$
(B) $\pi\delta\left(\omega - \frac{3\pi}{5}\right) - \pi\delta\left(\omega + \frac{3\pi}{5}\right)$
(C) $\sum_{l=-\infty}^{\infty} \pi\left[\delta\left(\omega - \frac{3\pi}{5} - 2\pi l\right) + \delta\left(\omega + \frac{3\pi}{5} - 2\pi l\right)\right]$
(D) $\sum_{l=-\infty}^{\infty} \pi\left[\delta\left(\omega - \frac{3\pi}{5} - 2\pi l\right) - \delta\left(\omega + \frac{3\pi}{5} - 2\pi l\right)\right]$

13. The solution to the differential equation $y'' - 2y' - 8y = f(t)$ with $y(0) = 1$ and $y'(0) = 0$, using the Laplace transform, is given by:

$$y(t) = Ae^{4t} * f(t) + Be^{-2t} * f(t) + Ce^{4t} + De^{-2t}.$$

where $*$ denotes convolution. What are the correct values of A, B, C, and D?
(A) $A = \frac{1}{6}, B = -\frac{1}{6}, C = \frac{1}{3}, D = \frac{2}{3}$
(B) $A = \frac{1}{3}, B = \frac{2}{3}, C = \frac{1}{6}, D = -\frac{1}{6}$
(C) $A = \frac{2}{3}, B = -\frac{1}{3}, C = \frac{1}{6}, D = \frac{1}{6}$
(D) $A = \frac{1}{6}, B = \frac{1}{6}, C = -\frac{1}{3}, D = \frac{2}{3}$

第14題到第17題需要詳明推導計算過程。如推導計算過程錯誤，將酌扣分數或不給分。

14. (7%) Magnetic resonance imaging is a medical imaging technique that uses macroscopic nuclear magnetization (M) of hydrogen in human tissues as its signal source. According to the well-known Bloch equation, the longitudinal magnetization $M_z(t)$ follows the differential equation

$$\frac{dM_z(t)}{dt} = \frac{M_0 - M_z(t)}{T_1},$$

where M_0 is the steady-state nuclear magnetization, and T_1 presents the longitudinal relaxation time, which is a tissue-dependent constant. Assume that an excitation pulse is applied to the magnetization right before $t = 0$ to remove M_z completely; in other words, $M_z(0) = 0$. Find $M_z(t)$ on the interval $(0, \infty)$.

15. (8%) Use separation of variables to find the product solution of $u(x, y)$, which satisfies

$$\frac{1}{y}\frac{\partial u}{\partial x} = \frac{1}{3x}\frac{\partial u}{\partial y}$$

16. (10%) Given that an 3D object is enclosed by a surface $f(x, y, z) = 0$ and the xy plane (i.e., $z = 0$), where $f(x, y, z) = x^2 + y^2 + z - 2$.

(1) (4%) Find the normal vector over the surface of this object. Note that the normal vector should be provided in the form of a unit vector here.

(2) (6%) What is the surface area of this object?

17. (10%) Given that z is a complex variable and $\bar{z}$ represents its complex conjugate.

(1) (2%) Sketch the set of points determined by $z\bar{z} = 2$ on the complex plane.

(2) (8%) Let C be the contour of your answer in (1) in a positive sense. Solve

$$\int_C \frac{e^{-z}}{z^2 + 1} dz$$