國立臺灣師範大學 114 學年度碩士班招生考試試題

科目：**數學基礎**　　　　　　　　　　　　　　適用系所：**資訊工程學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Some notations:

- A *vector* refers to a column vector with real entries, for example, $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathcal{R}^3$.
- Let $\mathcal{B}=\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}$ be a basis for $\mathcal{R}^n$. For each $\boldsymbol{v}$ in $\mathcal{R}^n$, the **coordinate vector** of $\boldsymbol{v}$ relative to $\mathcal{B}$, or called the **$\mathcal{B}$-coordinate vector** of $\boldsymbol{v}$, is denoted by $[\boldsymbol{v}]_{\mathcal{B}}$.
- The number of vectors in a basis for a nonzero subspace W of $\mathcal{R}^n$ is called the **dimension** of W and is denoted by dim W.

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1. (9 points) Let A be an $m \times n$ matrix with reduced row echelon form R and let $\boldsymbol{r}_i = R\boldsymbol{e}_i$ denote the i-th column of R, where $\boldsymbol{e}_i$ is the ith standard vector of $\mathcal{R}^n$ for $1 \leq i \leq n$. Determine the reduced row echelon form of each of the following matrices:
   (a) (3 points) $[A \quad \boldsymbol{0}]$, where $\boldsymbol{0}$ is a zero column
   (b) (3 points) $[\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \ldots \quad \boldsymbol{a}_k]$ for $k<n$, where $\boldsymbol{a}_i=A\boldsymbol{e}_i$.
   (c) (3 points) $[A \quad cA]$, where c is any scalar

2. (9 points) Suppose that A is a 4×4 matrix with no nonreal eigenvalues and exactly two real eigenvalues, 5 and -9. Let W_1 and W_2 be the eigenspaces of A corresponding to 5 and -9, respectively. Write all the possible characteristic polynomials of A that are consistent with the following information:
   (a) (4 points) $\dim W_1 = 3$
   (b) (5 points) $\dim W_2 = 2$

3. (10 points) Let $\mathcal{A}=\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a basis for $\mathcal{R}^n$. Let $\mathcal{B}=\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$, where $\boldsymbol{v}_i = \boldsymbol{u}_i + \boldsymbol{u}_{i+1} + \cdots + \boldsymbol{u}_n$ for $i = 1,2,\cdots,n$, also be a basis for $\mathcal{R}^n$. If $\boldsymbol{v}$ is a vector in $\mathcal{R}^n$ and $[\boldsymbol{v}]_{\mathcal{A}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, compute $[\boldsymbol{v}]_{\mathcal{B}}$.

4. (12 points) Let $\boldsymbol{u} = \begin{bmatrix} 4 \\ 1 \\ 3 \\ -1 \end{bmatrix}$ and a subspace $W = \text{Row}\begin{bmatrix} 2 & -2 & 3 & 4 \\ 1 & -1 & 1 & 1 \end{bmatrix}$ of $\mathcal{R}^4$.

   (a) (8 points) Find the orthogonal projection matrix P_W for subspace W.

   (b) (4 points) Find the unique vectors $\boldsymbol{w}$ in W that is closest to $\boldsymbol{u}$.

5. (10 points) Find a 3×3 matrix A having eigenvalues $-2, -2,$ and 4 with corresponding eigenvectors $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, and $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Notice: For the problems 6-9, simply put the answers on the answer sheet. No explanations are needed. We use $\mathbb{N}$ and $\mathbb{Z}$ to denote the set of positive integers and the set of integers, respectively.

6. (5 points) Let G be an undirected graph that has n vertices and m edges. Given that every vertex of G has degree 3, find n/m.

7. (15 points) The degree sequence of an undirected graph is the **nonincreasing** sequence of the vertex degrees. Let G be a **connected simple** undirected graph with five vertices. Given that the vertices of G have exactly three different degrees, please list all possible degree sequences of G.

8. (10 points) Let $x \in \mathbb{N}$ and $y \in \mathbb{Z}$. Find the **second smallest** x such that $314x + 159y = 26$.

9. (20 points) For $n \in \mathbb{N} \cup \{0\}$, let a_n be the number of binary relations (on an n-element set) that are symmetric and reflexive, and let b_n be the number of binary relations that are symmetric and NOT reflexive. By definition, we have $a_0 = 1$ and $b_0 = 0$.

   (a) (5 points) Find a_1 and b_1.

   (b) (10 points) Let f: $\mathbb{N} \to \mathbb{N}$ and g: $\mathbb{N} \to \mathbb{N}$ be two functions such that $b_n = f(n)b_{n-1} + g(n)a_{n-1}$ for $n > 0$. Find $f(n)$ and $g(n)$.

   (c) (5 points) Find a closed formula for b_n. Note: The operations allowed in a closed formula are arithmetic operations, taking powers, taking factorials, and the use of binomial coefficients.