國立中正大學
114學年度碩士班招生考試
試題
[第4節]

科目名稱	線性代數
系所組別	數學系
	數學系應用數學

—作答注意事項—

※作答前請先核對「試題」、「試卷」與「准考證」之系所組別、科目名稱是否相符。

1. 預備鈴響時即可入場，但至考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。
2. 考試開始鈴響時，即可開始作答；考試結束鈴響畢，應即停止作答。
3. 入場後於考試開始40分鐘內不得離場。
4. 全部答題均須在試卷（答案卷）作答區內完成。
5. 試卷作答限用藍色或黑色筆（含鉛筆）書寫。
6. 試題須隨試卷繳還。

NOTATION: In this test, all vector spaces are over $\mathbb{R}$. For a matrix $A \in M_{m\times n}$, let $L_A : \mathbb{R}^n \to \mathbb{R}^m$ denote left-multiplication transformation. Let $\mathbf{R}(L_A)$ denotes the range of L_A and $\mathbf{N}(L_A)$ denote the null space of L_A.

1. Let the matrix A be

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 & 9 \\ 1 & 4 & 7 & 10 & 13 \end{pmatrix}.$$

(a) Compute the reduced row echelon form of matrix A. (10pts)

(b) From the answer of part (a), find a basis of $\mathbf{N}(L_A)$ and a basis of $\mathbf{R}(L_A)$. (10pts)

(c) Orthogonally project the column vector

$$b = \begin{pmatrix} 5 \\ 10 \\ 0 \\ 5 \end{pmatrix}$$

onto $\mathbf{R}(L_A)$. This means find a vector in $\mathbf{R}(L_A)$ which is closest to the given vector. (10pts)

2. $k_1, k_2, \cdots, k_n \in \mathbb{R}^n$ are real numbers. Compute the following determinant. (10pts)

$$\begin{vmatrix} 1 & k_1 & k_1^2 & \cdots & k_1^{n-1} \\ 1 & k_2 & k_2^2 & \cdots & k_2^{n-1} \\ 1 & k_3 & k_3^2 & \cdots & k_3^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & k_n & k_n^2 & \cdots & k_n^{n-1} \end{vmatrix}$$

3. Let the matrix B be

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(a) Diagonalize B. This means you have to find an invertible matrix P and a diagonal matrix D such that $B = PDP^{-1}$. (10pts)

(b) For any $n \in \mathbb{N}$, compute the general formula for B^n. (10pts)

4. Proof the following:

(a) Let $A \in M_{n\times n}(\mathbb{R})$. Show that if $\{Av_1, Av_2, \cdots Av_k\}$ is linearly independent in $\mathbb{R}^n$, then $\{v_1, v_2, \cdots, v_k\}$ is linearly independent. (10pts)

(b) The rank of a matrix is defined as the dimension of its range. $\mathbf{rank}(A) = \dim(\mathbf{R}(L_A))$. Use part (a), show that for all $A, B \in M_{n\times n}(\mathbb{R})$, $\mathbf{rank}(AB) \leq \mathbf{rank}(B)$. (10pts)

5. In $\mathbb{R}^n$, the inner product is defined as

$$(x_1, x_2, \cdots, x_n) \cdot (y_1, y_2, \cdots, y_n) = \sum_{i=1}^{n} x_i y_i.$$

For any subspace $W \subset \mathbb{R}^n$, the orthogonal complement of W, denoted by $W^\perp$, is defined as

$$W^\perp = \{v | \forall w \in W, v \cdot w = 0\}.$$

(a) Show that $W^\perp$ is closed under addition and scalar multiplication. Therefore, it is a subspace. (5pts)

(b) Show that $W^\perp \cap W = \{0\}$. (5pts)

(c) Show that if $W_1, W_2 \subset \mathbb{R}^n$ are subspaces, we have $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$. (10pts)