國立中山大學114學年度
碩士班考試入學招生考試試題

科目名稱：線性代數【通訊所碩士班甲組】

—作答注意事項—

考試時間：100分鐘

- **考試開始鈴響前不得翻閱試題，並不得書寫、劃記、作答。**請先檢查答案卷（卡）之應考證號碼、桌角號碼、應試科目是否正確，如有不同立即請監試人員處理。
- **答案卷限用藍、黑色筆(含鉛筆)書寫、繪圖或標示**，可攜帶橡皮擦、無色透明無文字墊板、尺規、修正液（帶）、手錶(未附計算器者)。每人每節限使用一份答案卷，請衡酌作答。
- 答案卡請以2B鉛筆劃記，不可使用修正液（帶）塗改，未使用2B鉛筆、劃記太輕或污損致光學閱讀機無法辨識答案者，後果由考生自負。
- 答案卷（卡）應保持清潔完整，不得折疊、破壞或塗改應考證號碼及條碼，亦不得書寫考生姓名、應考證號碼或與答案無關之任何文字或符號。
- 可否使用計算機請依試題資訊內標註為準，如「可以」使用，廠牌、功能不拘，唯不得攜帶書籍、紙張（應考證不得做計算紙書寫）、具有通訊、記憶、傳輸或收發等功能之相關電子產品或其他有礙試場安寧、考試公平之各類器材入場。
- 試題及答案卷（卡）請務必繳回，未繳回者該科成績以零分計算。
- 試題採雙面列印，考生應注意試題頁數確實作答。
- 違規者依本校招生考試試場規則及違規處理辦法處理。

國立中山大學114學年度碩士班考試入學招生考試試題

科目名稱：線性代數【通訊所碩士班甲組】 **題號：437006**

※本科目依簡章規定「可以」使用計算機（廠牌、功能不拘）（選擇題） 共5頁第1頁

一、單選題 （每題5分，計分方式:不倒扣，答對得該題全部分數，答錯及未作答得零分）

1. (5%) Which of the following statement is **False**?
   (A) If A is a 3×4 matrix, then $\mathbf{Ax} = \mathbf{0}$ has infinitely many solutions.
   (B) If a system of equations has fewer equations than unknowns, then it has infinitely many solutions.
   (C) If $\mathbf{A}$ in an $n \times m$ matrix and $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{x}$ could be an nonzero vector.
   (D) If a square matrix has two equal rows, then it is not invertible.
   (E) If a square matrix has two equal columns, then it is not invertible.

2. (5%) Which of the following statement is **False**?
   (A) There exists a 2×2 matrix $\mathbf{A}$ such that $\mathbf{A}^2 = \mathbf{I}$ but $\mathbf{A}^4 \neq \mathbf{I}$.
   (B) There exists a 2×2 matrix $\mathbf{A}$ such that $\mathbf{A}^3 = \mathbf{I}$ but $\mathbf{A} \neq \mathbf{I}$.
   (C) There exists a 2×2 matrix $\mathbf{A}$ such that $\mathbf{A}^4 = \mathbf{I}$ but $\mathbf{A} \neq \mathbf{I}$.
   (D) There exists a 2×2 matrix $\mathbf{A}$ such that rank($\mathbf{A}$) = 0.
   (E) There exists a 2×2 matrix $\mathbf{A}$ such that rank($\mathbf{A}$) = 2.

3. (5%) Which of the following statement is **False**?
   (A) Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. If $\mathbf{v}$ is in $\ker(\mathbf{B})$, then $\mathbf{v}$ is in $\ker(\mathbf{AB})$.
   (B) If $\mathbf{v}$ and $\mathbf{w}$ are in $\mathrm{im}(\mathbf{A})$, then $2\mathbf{v} - 7\mathbf{w}$ is in $\mathrm{im}(\mathbf{A})$ too.
   (C) If $\mathbf{x} = \mathbf{v}$ and $\mathbf{x} = \mathbf{w}$ are two solutions to $\mathbf{Ax} = \mathbf{b}$, then $\mathbf{x} = \mathbf{v} + \mathbf{w}$ is a solution too.
   (D) If $\mathbf{Av} = \mathbf{Aw}$, then $\mathbf{v} - \mathbf{w}$ is in $\ker(\mathbf{A})$.
   (E) Let $\mathbf{A}$ be an $n \times m$ matrix. Then $\mathrm{im}(\mathbf{A})$ is in a subspace of $\mathbb{R}^n$.

4. (5%) Which of the following statement is **False**?
   (A) Let $\mathbf{A}$ be an $n \times n$ matrix. If $\mathbf{A}^{-1} = A^{\mathrm{T}}$, then the columns of $\mathbf{A}$ form an orthonormal basis of $\mathbb{R}^n$.
   (B) If $\mathbf{A}$ is an orthogonal $n \times n$ matrix, then the least-squares solution to $\mathbf{Ax} = \mathbf{b}$ is unique and the solution is $\mathbf{A}^{\mathrm{T}}\mathbf{b}$.
   (C) Let $\mathbf{A}$ be an $n \times m$ matrix. If the least-squares solution to $\mathbf{Ax} = \mathbf{b}$ is unique, then $\ker(\mathbf{A}) = \{\mathbf{0}\}$.
   (D) Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. If $\mathbf{A}$ is similar to $\mathbf{B}$, then $\mathbf{B}$ is similar to $\mathbf{A}$.
   (E) Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. If $\mathbf{A}$ is similar to $\mathbf{B}$, then $\mathbf{A} = \mathbf{B}$.

5. (5%) Which of the following statement is **False**?
   (A) Let $\mathbf{A}$ be an $n \times n$ matrix. If $\mathrm{rank}(\mathbf{A}) \neq n$, then 0 is an eigenvalue of $\mathbf{A}$.
   (B) If a matrix has no eigenvalues, then it has no eigenvectors.
   (C) Let $\mathbf{A}$ be an $n \times n$ matrix. Then $\det(\mathbf{A}^{\mathrm{T}}) = \det(\mathbf{A})$.
   (D) Let $\mathbf{A}$ be an $n \times n$ matrix. Then $\det(-\mathbf{A}) = -\det(\mathbf{A})$.
   (E) Let $\mathbf{B}$ be an $(n-1) \times (n-1)$ matrix and $\mathbf{A}$ be the $n \times n$ matrix $\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}$ (where the $\mathbf{0}$ entries represent zero matrices of the appropriate size). Then $\det(\mathbf{A}) = \det(\mathbf{B})$.

6. (5%) Let $\mathbf{A}$ be a 3×3 matrix and let
   $$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$
   Suppose that $\mathbf{Av} = -\mathbf{v}$ and $\mathbf{Aw} = \mathbf{2w}$. The vector $\mathbf{A}^5 \begin{bmatrix} -1 \\ 8 \\ -9 \end{bmatrix}$ is
   (A) $[-131 \quad 58 \quad -189]^{\mathrm{T}}$.

(B) $[58 \quad -131 \quad -189]^{\mathrm{T}}$.
(C) $[-131 \quad 189 \quad 58]^{\mathrm{T}}$.
(D) $[131 \quad 58 \quad -189]^{\mathrm{T}}$.
(E) $[0 \quad 0 \quad 0]^{\mathrm{T}}$.

7. (5%) Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices with real entries. Assume that $\mathbf{A}$ and $\mathbf{B}$ are not invertible, but $\mathbf{A}+\mathbf{B}$ is invertible. Then $\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B} =$
(A) $\mathbf{0}$.
(B) $\mathbf{I}$.
(C) $(\mathbf{A}+\mathbf{B})^{-1}\mathbf{AB}$.
(D) $\mathbf{AB}(\mathbf{A}+\mathbf{B})^{-1}$.
(E) $\mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}$.

8. (5%) Let $\mathbf{J}$ be the $n \times n$ matrix of all 1's, and consider $\mathbf{A} = (a-b)\mathbf{I} + b\mathbf{J}$. Then the eigenvalues of $\mathbf{A}$ are
(A) $a+b$, and $a+(n-1)b$.
(B) $a-nb$, and $a+nb$.
(C) $a-b$, and $a+(n-1)b$.
(D) $a-2b$, and $a+nb$.
(E) $a+b$, and $a-(n-1)b$.

9. (5%) Let $\mathbf{A} = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} & \frac{3}{7} \\ \frac{3}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{3}{7} & \frac{3}{7} & \frac{1}{7} \end{bmatrix}$ be 3×3 matrix. Then $\lim_{n\to\infty} A^n$

(A) $\begin{bmatrix} \frac{1}{7} & \frac{3}{7} & \frac{3}{7} \\ \frac{3}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{3}{7} & \frac{3}{7} & \frac{1}{7} \end{bmatrix}$.

(B) $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

(C) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(E) $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$.

10. (5%) The determinant of

$$\mathbf{A} = \begin{bmatrix} 3a & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 7 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

is

(A) 0.
(B) $14a$.
(C) $-14a$.
(D) $12a$.
(E) $-12a$.

11. (5%) Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

Which of the following statement is **False**?

(A) The row space of **A** has a dimension of 2.
(B) The column space of **A** has a dimension of 2.
(C) The null space of **A** has a dimension of 2.
(D) There are exact one solution of $\mathbf{Ax} = \mathbf{0}$
(E) There are two nonzero eigenvalues of $\mathbf{A}^T\mathbf{A}$.

12. (5%) Suppose **A** is a 5×3 matrix and **Ax** is never zero except when **x** is a zero vector. Which of the following statement is **False**?

(A) **A** has full rank
(B) The column of **A** has a dimension of 3.
(C) The row space of **A** has a dimension of 5.
(D) $\mathbf{A}^T\mathbf{Ax}$ is also never zero except when **x** is a zero vector.
(E) $\mathbf{A}^T\mathbf{A}$ is non-singular.

13. (5%) Suppose **A** is a $M \times M$ matrix with rank r and $\mathbf{A}^2 = \mathbf{A}$

Which of the following statement is **False**?

(A) -1 is a possible eigenvalue of A.
(B) $\mathbf{A}^n = \mathbf{A}$, for any positive integer n.
(C) $(\mathbf{I} - \mathbf{A})^2 = \mathbf{I} - \mathbf{A}$.
(D) $\text{tr}(\mathbf{A}) = r$.
(E) If $r = M$, **A** should be an identity matrix.

14. (5%) Suppose $adf \neq 0$, which of the following matrix is an inverse of $\mathbf{A} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$

(A) $\frac{1}{adf}\begin{bmatrix} df & -bf & be + cd \\ 0 & af & ae \\ 0 & 0 & ad \end{bmatrix}$.

(B) $\frac{1}{adf}\begin{bmatrix} df & -bf & be - cd \\ 0 & ae & -af \\ 0 & 0 & ad \end{bmatrix}$.

(C) $\frac{1}{adf}\begin{bmatrix} df & bf & be-cd \\ 0 & af & ae \\ 0 & 0 & ad \end{bmatrix}$.

(D) $\frac{1}{adf}\begin{bmatrix} df & bf & bd-ce \\ 0 & af & -ae \\ 0 & 0 & ad \end{bmatrix}$.

(E) $\frac{1}{adf}\begin{bmatrix} df & -bf & be-cd \\ 0 & af & -ae \\ 0 & 0 & ad \end{bmatrix}$.

15. (5%) Consider a 3×3 matrix **A**, which is symmetric positive definite. Let λ_i and $\boldsymbol{u}_i$ be an eigen-value and eigen-vector of **A**, i.e., $\mathbf{A}\mathbf{u}_i = \lambda_i \mathbf{u}_i$, with $\lambda_1 > \lambda_2 > \lambda_3$. Suppose $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$. Which of the following statement is **False**?
(A) $\lambda_i > 0$, for $i = 1,2,3$.
(B) $\mathbf{A}^H\mathbf{A} = \lambda_1\mathbf{u}_1\mathbf{u}_1^H + \lambda_2\mathbf{u}_2\mathbf{u}_2^H + \lambda_3\mathbf{u}_3\mathbf{u}_3^H$.
(C) $\mathbf{x}^H\mathbf{x} = |c_1|^2 + |c_2|^2 + |c_3|^2$.
(D) $\mathbf{x}^H\mathbf{A}\mathbf{x} = |c_1|^2\lambda_1 + |c_2|^2\lambda_2 + |c_3|^2\lambda_3$.
(E) The ratio $\mathbf{x}^H\mathbf{A}\mathbf{x}/\mathbf{x}^H\mathbf{x}$ is maximized when $c_2 = c_3 = 0$.

16. (5%) Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Which of the following statement is **False**?
(A) $\mathbf{A}^{-1} = \mathbf{A}^T$.
(B) $(\mathbf{A}^2)^{-1} = \mathbf{A}^2$.
(C) Eigenvalues of **A** are $1, 1, -1, -1$.
(D) Determinant of **A+I** is 0.
(E) Determinant of **A+2I** is 15.

17. (5%) Consider the following equations:

$$\begin{cases} x + y + 2z = k \\ 3x + y + 7z = 3k + 4 \\ 4x + 2y + (k+7)z = 5k \end{cases}$$

For which value of k the system has **NO** solution.
(A) $k = -1$
(B) $k = 0$
(C) $k = 1$
(D) $k = 2$
(E) $k = 4$

18. (5%) Let $\mathcal{S} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}.$$

Take Gram-Schmidt to find the orthonormal basis of the space $\mathcal{S}$ with the order $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ accordinly, and denote the basis vectors as $\mathbf{u}_1, \mathbf{u}_2, \ldots$ Which of the following statement is **False**?

(A) $\mathbf{u}_1 = \frac{1}{2}[1 \quad 1 \quad 1 \quad 1]^T$.

(B) $\mathbf{u}_2 = \frac{1}{2}[-1 \quad 1 \quad 1 \quad -1]^T$.

(C) $\mathbf{u}_3 = \frac{1}{2}[1 \quad -1 \quad 1 \quad -1]^T$.

(D) Dimension of $\mathcal{S}$ is 3.

(E) None of above.

19. (5%) Take QR decomposition on the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \mathbf{QR},$$

where **Q** is unitary and **R** is upper triangular. Which of the following is the matrix **R**?

(A) $\mathbf{R} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & 4 \end{bmatrix}$.

(B) $\mathbf{R} = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$.

(C) $\mathbf{R} = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

(D) $\mathbf{R} = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -1 \\ 0 & 0 & 2 \end{bmatrix}$.

(E) $\mathbf{R} = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

20. (5%) Which of the following vector is the orthogonal projection of $[1 \quad -2 \quad 2 \quad 1]^T$ onto the column space of $\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$?

(A) $[2 \quad -1 \quad 1 \quad 0]^T$.

(B) $[1 \quad 0 \quad 2 \quad -1]^T$.

(C) $[0 \quad 1 \quad -1 \quad 2]^T$.

(D) $[1 \quad -2 \quad 0 \quad 1]^T$.

(E) $[-1 \quad 2 \quad 1 \quad 0]^T$.