國立中山大學114學年度
碩士班考試入學招生考試試題

科目名稱：機率【通訊所碩士班甲組】

—作答注意事項—

考試時間：100分鐘

- 考試開始鈴響前不得翻閱試題，並不得書寫、劃記、作答。請先檢查答案卷（卡）之應考證號碼、桌角號碼、應試科目是否正確，如有不同立即請監試人員處理。
- 答案卷限用藍、黑色筆(含鉛筆)書寫、繪圖或標示，可攜帶橡皮擦、無色透明無文字墊板、尺規、修正液（帶）、手錶(未附計算器者)。每人每節限使用一份答案卷，請衡酌作答。
- 答案卡請以2B鉛筆劃記，不可使用修正液（帶）塗改，未使用2B鉛筆、劃記太輕或污損致光學閱讀機無法辨識答案者，後果由考生自負。
- 答案卷（卡）應保持清潔完整，不得折疊、破壞或塗改應考證號碼及條碼，亦不得書寫考生姓名、應考證號碼或與答案無關之任何文字或符號。
- 可否使用計算機請依試題資訊內標註為準，如「可以」使用，廠牌、功能不拘，唯不得攜帶書籍、紙張（應考證不得做計算紙書寫）、具有通訊、記憶、傳輸或收發等功能之相關電子產品或其他有礙試場安寧、考試公平之各類器材入場。
- 試題及答案卷（卡）請務必繳回，未繳回者該科成績以零分計算。
- 試題採雙面列印，考生應注意試題頁數確實作答。
- 違規者依本校招生考試試場規則及違規處理辦法處理。

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※本科目依簡章規定「可以」使用計算機（廠牌、功能不拘）（混合題）　　共4頁第1頁

一、選擇題(單選，計分方式:不倒扣，答對得該題全部分數，答錯及未作答得零分)

1. (5%) For $n \in \mathbb{Z}^+$, $\sum_{k=0}^{n} k \binom{n}{k} =$

(A) 0 .
(B) 2^n .
(C) $n\,2^{n-1}$.
(D) $n\,2^n$.
(E) $n\,2^{n+1}$.

2. (5%) For $n \in \mathbb{Z}^+$, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} =$

(A) 0 .
(B) 2^{n-1} .
(C) 2^n .
(D) $n\,2^{n-1}$.
(E) $n\,2^n$.

3. (5%) What is the total number of possible solutions for $x_1 + x_2 + x_3 = 10$, where $x_1, x_2, x_3 \in \{0\} \cup \mathbb{Z}^+$?.

(A) 36
(B) 45
(C) 66
(D) 120
(E) 286

4. (5%) How many 11-letter words out of "examination"?

(A) 40 320
(B) 4 989 600
(C) 13 305 600
(D) 39 916 800
(E) None of the above

5. (5%) Which of the following is not a type of random variables?

(A) Continuous random variables
(B) Discrete random variables
(C) Hybrid random variables
(D) Mixed random variables
(E) None of the above

6. (5%) Given that $a, b \in \mathbb{R}$, which of the following is true about the CDF of any random variable X?
   (A) $F_X(a) > F_X(b)$ when $a > b$
   (B) $F_X(a) \geq F_X(b)$ when $a \geq b$
   (C) $F_X(a) = F_X(b)$ if and only if $a = b$
   (D) $F_X(a + b) = F_X(a) + F_X(b)$ for any a and b
   (E) None of the above

7. (5%) If $H(x)$ and $\delta(x)$ are, respectively, the Heaviside step function and the Dirac delta function evaluated at x (where $x \in \mathbb{R}$), then $\int_{-\infty}^{x} H(\phi)\, d\phi =$
   (A) 1
   (B) $\delta(x)$
   (C) $x\,\delta(x)$
   (D) $H(x)$
   (E) $x\,H(x)$

8. (5%) Let X and Y be two random variables such that they are linearly dependent. Which of the following about ρ_{XY}, the correlation coefficient of X and Y, is correct?
   (A) $\rho_{XY} = -1$
   (B) $\rho_{XY} = 0$
   (C) $\rho_{XY} = 1$
   (D) $\rho_{XY} = -1$ or $\rho_{XY} = 1$
   (E) None of the above

9. (5%) Given that X is a Poisson random variable with mean μ, find $M_X(t)$, the moment-generating function of X.
   (A) $M_X(t) = \exp[\mu\,(e^t - 1)]$
   (B) $M_X(t) = \exp[\mu\,(e^t + 1)]$
   (C) $M_X(t) = \exp(\mu t - 1)$
   (D) $M_X(t) = \exp(\mu t + 1)$
   (E) None of the above

10. (5%) Let X be a discrete random variable such that $f_X(x) = \frac{3}{4}\left(\frac{1}{4}\right)^{x-1}$, where $x \in \mathbb{Z}^+$. If $Y = X^2$, find $f_Y(y)$, where $y \in \{1, 4, 9, \dots\}$.

(A) $f_Y(y) = \frac{3}{4}\left(\frac{1}{4}\right)^{y-1}$

(B) $f_Y(y) = \frac{3}{4}\left(\frac{1}{4}\right)^{y^2-1}$

(C) $f_Y(y) = \frac{3}{4}\left(\frac{1}{4}\right)^{-\sqrt{y}-1}$

(D) $f_Y(y) = \frac{3}{4}\left(\frac{1}{4}\right)^{\sqrt{y}-1}$

(E) None of the above

二、問答計算題:

1. (10%) Consider an experiment in which a fair coin is tossed three times.

(a). (3%) What is the sample space (Ω) of all possible outcomes for this experiment?

(b). (4%) Define the event E_i as the outcomes where exactly i tosses result in "heads," where i = 0,1,2,3. For each i=0,1,2,3, how many sample points does E_i contain?

(c). (3%) Define event F as an event in which at least two of the tosses yield "heads."

2. (10%) Consider two independent random variables X and N, where X is uniformly distributed on $(0,2\pi)$ and N follows a Gaussian distribution with probability density function (PDF) $f(N) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{N^2}{2\sigma^2}\right)$. Let Y=X+N. Determine the conditional PDF $f(X \mid Y)$.

3. (15%) Let N be a positive integer-valued random variable with the cumulative distribution function (CDF) $F_N(n) = \frac{n}{n+1}$ for each integer $n \geq 1$.

(a). (5%) Use the CDF $F_N(n)$ to demonstrate that N is a positive random variable.

(b). (5%) Determine the probability mass function (PMF) of N, i.e., $P_N(n)$.

(c). (5%) Compute the expected value of N, i.e., $E\{N\}$.

4. (15%) Consider an experiment in which a fair coin is tossed n times. Let S_N represent the total number of "heads (=1)" obtained:

$$S_N = B_1 + B_2 + \cdots + B_N,$$

where B_i are independent binary random variables with $P(B_i = 1) = P(B_i = 0) = 0.5$ for all $i = 0,1,2,\ldots,N$.

(a). (5%) Compute the moment generating function (MGF) of S_N, denoted as $M_{S_N}(t)$.

Hint: For a random variable X, its MGF $M_X(t)$ is defined by $M_X(t) = E\left[e^{tX}\right]$.

(b). (10%) The Chernoff bound can be given as

$$P(S_N \geq N\beta) \leq \min_{t\geq 0}\left\{e^{-tN\beta} M_{S_N}(t)\right\}.$$

Determine the optimal value of t and show that

$$P(S_N \geq N\beta) \leq 2^{-N(1-\mathcal{H}(\beta))},$$

where $\mathcal{H}(\beta) = -\beta\log_2\beta - (1-\beta)\log_2(1-\beta)$.

Hint: Minimizing $e^{-tN\beta} M_{S_N}(t)$ is equal to minimizing $\ln\left(e^{-tN\beta} M_{S_N}(t)\right)$.