國立中山大學114學年度
碩士班考試入學招生考試試題

科目名稱：工程數學乙【電機系碩士班乙組選考】

—作答注意事項—

考試時間：100分鐘

- **考試開始鈴響前不得翻閱試題，並不得書寫、劃記、作答。**請先檢查答案卷（卡）之應考證號碼、桌角號碼、應試科目是否正確，如有不同立即請監試人員處理。
- **答案卷限用藍、黑色筆(含鉛筆)書寫、繪圖或標示**，可攜帶橡皮擦、無色透明無文字墊板、尺規、修正液（帶）、手錶(未附計算器者)。每人每節限使用一份答案卷，請衡酌作答。
- 答案卡請以2B鉛筆劃記，不可使用修正液（帶）塗改，未使用2B鉛筆、劃記太輕或污損致光學閱讀機無法辨識答案者，後果由考生自負。
- 答案卷（卡）應保持清潔完整，不得折疊、破壞或塗改應考證號碼及條碼，亦不得書寫考生姓名、應考證號碼或與答案無關之任何文字或符號。
- 可否使用計算機請依試題資訊內標註為準，如「可以」使用，廠牌、功能不拘，唯不得攜帶書籍、紙張（應考證不得做計算紙書寫）、具有通訊、記憶、傳輸或收發等功能之相關電子產品或其他有礙試場安寧、考試公平之各類器材入場。
- 試題及答案卷（卡）請務必繳回，未繳回者該科成績以零分計算。
- 試題採雙面列印，考生應注意試題頁數確實作答。
- 違規者依本校招生考試試場規則及違規處理辦法處理。

國立中山大學114學年度碩士班考試入學招生考試試題

科目名稱：工程數學乙【電機系碩士班乙組選考】 題號：431001

※本科目依簡章規定「可以」使用計算機（廠牌、功能不拘）（混合題）

第1-10題為複選題，每題5分，總分50分。每題有5個選項，其中至少有1個是正確答案，答錯1個選項者，得3分；答錯2個選項者，得1分；答錯多於2個選項或未作答者，該題以零分計算。

1. Consider the linear system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] \in \mathbb{R}^{4\times 3}$, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are column vectors of $\mathbf{A}$, and $\mathbf{b} \in \mathbb{R}^4$ is a non-zero vector. Suppose

$$\mathbf{a}_1 + 2\mathbf{a}_2 = 3\mathbf{a}_3,\ \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{b},\ \mathbf{a}_2 + 2\mathbf{a}_3 \neq \mathbf{0}.$$

Which of the following statements are true?
(A) The linear system has a finite number of solutions.
(B) $\text{rank}([\mathbf{A}, \mathbf{b}])$ is equal to $\text{rank}(\mathbf{A})$.
(C) The rank of $\mathbf{A}$ is less than or equal to 2.
(D) $\mathbf{x} = [1, 2, 3]^T$ is a solution of the linear system.
(E) The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are linearly dependent.

2. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Which of the following statements are true?
(A) $\text{rank}(\mathbf{A}) = 2$.
(B) $\text{trace}(\mathbf{A}) = 7$.
(C) $\det(\mathbf{A}) \neq 0$.
(D) 3 is an eigenvalue of $\mathbf{A}$.
(E) All eigenvalues of $\mathbf{A}$ are non-negative.

3. Let

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] = \mathbf{QR} = \mathbf{Q}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix},$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are column vectors of $\mathbf{A} \in \mathbb{R}^{3\times 3}$, and $\mathbf{Q} \in \mathbb{R}^{3\times 3}$ is an orthogonal matrix, i.e., $\mathbf{Q}^T\mathbf{Q} = \mathbf{QQ}^T$ is the identity matrix. Which of the following statements are true?

(A) $\mathbf{a}_3^T\mathbf{a}_1 = 3$, (B) $\mathbf{a}_3^T\mathbf{a}_2 = 24$, (C) $\mathbf{a}_3^T\mathbf{a}_3 = 70$, (D) $\mathbf{a}_2^T\mathbf{a}_1 = 3$, (E) $\mathbf{a}_2^T\mathbf{a}_2 = 30$.

4. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3],$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are column vectors of $\mathbf{A}$. Let $\mathbf{q}_3$ be the orthogonal projection of $\mathbf{a}_3$ onto $\text{span}(\mathbf{a}_1, \mathbf{a}_2)$, and $\mathbf{q}_3 = [q_{31}, q_{32}, q_{33}, q_{34}]^T$. Which of the following statements are true?

(A) $q_{31} = 4$, (B) $q_{32} = 4/3$, (C) $q_{33} = 10/3$, (D) $q_{34} = 22/3$, (E) $\mathbf{a}_1^T\mathbf{q}_3 = 20$.

5. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{A}\mathbf{A}^T = \mathbf{I}$, and $\mathbf{A}$ is a non-zero matrix. Which of the following statements are true?
   (A) $\text{trace}(\mathbf{A}) \neq 0$.
   (B) $\det(\mathbf{A}) = 1$.
   (C) $\text{rank}(\mathbf{A}) = n$.
   (D) If $n = 3$, then $\text{trace}(\mathbf{A}) \leq 3$.
   (E) All eigenvalues of $\mathbf{A}$ are real numbers.

6. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, where $R(\mathbf{A})$ denotes the column space of $\mathbf{A}$, $N(\mathbf{A})$ denotes the null space of $\mathbf{A}$, and $\dim(S)$ denotes the dimension of a subspace S. Which of the following statements are true?
   (A) If $\mathbf{A}$ has linearly independent columns, then $\mathbf{A}\mathbf{A}^T$ is nonsingular.
   (B) $\text{rank}(\mathbf{A}^T) + \dim(N(\mathbf{A}^T)) = m$.
   (C) $R(\mathbf{A}\mathbf{A}^T) = R(\mathbf{A}^T\mathbf{A})$.
   (D) If $\mathbf{A}\mathbf{A}^T$ is nonsingular, then $\mathbf{A}^T\mathbf{A}$ is also nonsingular.
   (E) It is possible for a matrix $\mathbf{A}$ to have $[3, 4, 5]^T$ in $R(\mathbf{A})$ and $[1, -1, 2]^T$ in $N(\mathbf{A}^T)$.

7. Let $\mathbf{A} \in \mathbb{R}^{3\times 3}$ with $\text{rank}(\mathbf{A}^2) = \text{rank}(\mathbf{A}) - 1$, and $\mathbf{A}$ has at least two distinct eigenvalues. Which of the following statements are true?
   (A) $\mathbf{A}$ is diagonalizable.
   (B) $\mathbf{A}$ is not symmetric.
   (C) $\text{rank}(\mathbf{A}^2) = 1$.
   (D) $\text{trace}(\mathbf{A}) \neq 0$.
   (E) $\mathbf{A}$ is a nonsingular matrix.

8. Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in an inner product space V, and $\mathbf{p}$ is the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$. Define $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$. Which of the following statements are true?
   (A) $\|\mathbf{u}\| \geq \|\mathbf{u} - \mathbf{p}\|$.
   (B) $\|\mathbf{p}\| \geq \|\mathbf{u} - \mathbf{p}\|$.
   (C) $\mathbf{p} = \mathbf{u}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
   (D) $\|\mathbf{u} - \mathbf{p}\|^2 = \|\mathbf{u}\|^2 - \|\mathbf{p}\|^2$.
   (E) $\|\mathbf{u} - \mathbf{p}\| \leq \|\mathbf{u} - \alpha\mathbf{v}\|$ for any $\alpha \in \mathbb{R}$.

9. Let P^2 be the space of all real-valued polynomials of degree at most 2, with the inner product defined as
$$\langle f, g\rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1)$$
   where $f(x), g(x) \in P^2$. Define $\|f\| = \sqrt{\langle f, f\rangle}$. Consider the orthogonal set of polynomials $\{u_1, u_2, u_3\}$ in P^2 with respect to this inner product, where
$$u_1(x) = 1,\ u_2(x) = x,\ u_3(x) = x^2 - 2/3.$$
   Suppose $p(x) \in P^2$ is a polynomial such that $\langle p, u_1\rangle = 6$, $\langle p, u_2\rangle = -6$, $\langle p, u_3\rangle = 4$. Let the orthogonal projection of $p(x)$ onto $\text{span}(u_1, u_2)$ be denoted by $q(x)$. Which of the following statements are true?
   (A) $\langle u_3, u_3\rangle = 2/3$.
   (B) $q(x) = 2u_1(x) - 3u_2(x)$.
   (C) $\|q(x)\|^2 = 30$.
   (D) $p(x) = 6u_1(x) - 6u_2(x) + 4u_3(x)$.
   (E) $\|p(x)\|^2 = 572/3$.

10. Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator defined as

$$L(\mathbf{x}) = \begin{bmatrix} d_2x_3 - d_3x_2 \\ d_3x_1 - d_1x_3 \\ d_1x_2 - d_2x_1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

where d_1, d_2, d_3 are given scalars. Let $\mathbf{A}$ be the matrix representation of L with respect to the standard basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$, where $\mathbf{e}_1 = [1,0,0]^T$, $\mathbf{e}_2 = [0,1,0]^T$, and $\mathbf{e}_3 = [0,0,1]^T$. Denote the last two rows of $\mathbf{A}$ by $[a_{21}, a_{22}, a_{23}]$ and $[a_{31}, a_{32}, a_{33}]$, respectively. Which of the following statements are true?

(A) $a_{21} = d_2$, (B) $a_{22} = d_1$, (C) $a_{31} = -d_3$, (D) $a_{32} = d_1$, (E) $a_{33} = 0$.

第 11 -14 題需要詳明推導計算過程。如推導計算過程錯誤，將酌扣分數或不給分。

11. (12%) Magnetic resonance imaging is a medical imaging technique that uses macroscopic nuclear magnetization (M) of hydrogen in human tissues as its signal source. According to the well-known Bloch equation, the longitudinal magnetization $M_z(t)$ follows the differential equation

$$\frac{dM_z(t)}{dt} = \frac{M_0 - M_z(t)}{T_1},$$

where M_0 is the steady-state nuclear magnetization, and T_1 presents the longitudinal relaxation time, which is a tissue-dependent constant. Assume that an excitation pulse is applied to the magnetization right before $t = 0$ to remove M_z completely; in other words, $M_z(0) = 0$. Find $M_z(t)$ on the interval $(0, \infty)$.

12. (12%) Find the general solution of $y(x)$ for

$$\frac{dy^2}{dx^2} + \frac{dy}{dx} = e^x + e^{-x}$$

13. (12%) Find the general solution of $x(t)$ and $y(t)$ to satisfy the following equations.

$$\begin{cases} \dfrac{dx}{dt} = -6x + 2y \\ \dfrac{dy}{dt} = -3x + y \end{cases}$$

14. (1) (12%) Find the power series solution of $(x^2 - 1)\frac{dy^2}{dx^2} + 2x\frac{dy}{dx} - 2y = 0$ about the point $x = 0$.

(2) (2%) What is the radius of convergence for your solution in (1)?