國立中山大學114學年度
碩士班考試入學招生考試試題

科目名稱：機率與統計【應數系碩士班甲組】

—作答注意事項—

考試時間：100分鐘

- **考試開始鈴響前不得翻閱試題，並不得書寫、劃記、作答。**請先檢查答案卷（卡）之應考證號碼、桌角號碼、應試科目是否正確，如有不同立即請監試人員處理。
- **答案卷限用藍、黑色筆(含鉛筆)書寫、繪圖或標示**，可攜帶橡皮擦、無色透明無文字墊板、尺規、修正液（帶）、手錶(未附計算器者)。每人每節限使用一份答案卷，請衡酌作答。
- 答案卡請以2B鉛筆劃記，不可使用修正液（帶）塗改，未使用2B鉛筆、劃記太輕或污損致光學閱讀機無法辨識答案者，後果由考生自負。
- 答案卷（卡）應保持清潔完整，不得折疊、破壞或塗改應考證號碼及條碼，亦不得書寫考生姓名、應考證號碼或與答案無關之任何文字或符號。
- 可否使用計算機請依試題資訊內標註為準，如「可以」使用，廠牌、功能不拘，唯不得攜帶書籍、紙張（應考證不得做計算紙書寫）、具有通訊、記憶、傳輸或收發等功能之相關電子產品或其他有礙試場安寧、考試公平之各類器材入場。
- **試題及答案卷（卡）請務必繳回，未繳回者該科成績以零分計算。**
- **試題採雙面列印，考生應注意試題頁數確實作答。**
- **違規者依本校招生考試試場規則及違規處理辦法處理。**

- Please **answer the questions in order** and **write down the question number** for **each** question. If you are not able to answer the question, leave it blank.
- Notations: i.i.d., independent and identically distributed.

1. (20%) Let $X_1, \ldots, X_n$ be i.i.d. random samples from a population with finite mean μ and variance $\sigma^2 > 0$. Define the sample mean and variance as $\bar{X}_n = \frac{1}{n}\Sigma_{i=1}^n X_i$ and $S_n^2 = \frac{1}{n-1}\Sigma_{i=1}^n (X_i - \bar{X}_n)^2$, respectively.

   (a) (10%) Find $E(\bar{X}_n)$ and $\text{Var}(\bar{X}_n)$.
   (b) (10%) Show that S_n^2 is an unbiased estimator for σ^2.

2. (10%) Let X be a nonnegative continuous random variable. Show that

$$E(X^2) = 2\int_0^\infty tP(X > t)dt.$$

3. (25%) Let $X_1, \ldots, X_n$ be i.i.d. uniform random samples on the interval $[0, \theta]$.

   (a) (5%) Find the maximum likelihood estimator for θ and denote it as θ_n.
   (b) (5%) Find the asymptotic distribution of $n(\theta - \theta_n)$.
   (c) (5%) Show that the maximum likelihood estimator θ_n is a consistent estimator for θ.
   (d) (10%) Find an approximate level α test for testing the null hypothesis $H_0: \theta \le \theta_0$ against the alternative hypothesis $H_1: \theta > \theta_0$, where $\theta_0 > 0$ is some known constant, based on the asymptotic distribution of $n(\theta - \theta_n)$ in (b). Note that you have to justify the test you found is an approximately level α test.

4. (25%) Let X and Y be i.i.d. standard normal random variables, and $(X_1, Y_1), \ldots, (X_n, Y_n)$ be i.i.d. random samples distributed as (X, Y). Define the probability $p = P(X^2 + Y^2 \le 1)$.

   (a) (10%) Find p.
   (b) (5%) Find an estimator for p based on the samples $(X_1, Y_1), \ldots, (X_n, Y_n)$.
   (c) (10%) Establish asymptotic normality for the estimator you found in (b).

5. (20%) Assume that X and Y be are conditionally independent given W. The conditional survival functions of X and Y given $W = w$ are defined as $P(X > x \mid W = w) = e^{-\lambda_1 w x}$, $x > 0, \lambda_1 > 0$ and $P(Y > y \mid W = w) = e^{-\lambda_2 w y}$, $y > 0, \lambda_2 > 0$ respectively. Let W be a one-parameter gamma random variable with the probability density function

$$f_W(w) = \frac{w^{\frac{1}{\theta}-1} e^{-\frac{w}{\theta}}}{\Gamma\left(\frac{1}{\theta}\right)\theta^{\frac{1}{\theta}}}, \qquad w > 0, \theta > 0,$$

   where Γ is the gamma function.

   (a) (10%) Find the joint survival function $S(x, y) = P(X > x, Y > y)$, $x > 0, y > 0$.
   (b) (10%) Find the function $C: [0,1]^2 \mapsto [0,1]$ satisfying $S(x, y) = C\{S_X(x), S_Y(y)\}$, where $S_X(x) = P(X > x)$ and $S_Y(y) = P(Y > y)$ are the marginal survival functions of X and Y, respectively. **Hint:** If $f^{-1}(u)$ is the inverse function of $f(x)$, one has $f\left(f^{-1}(u)\right) = u$.

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