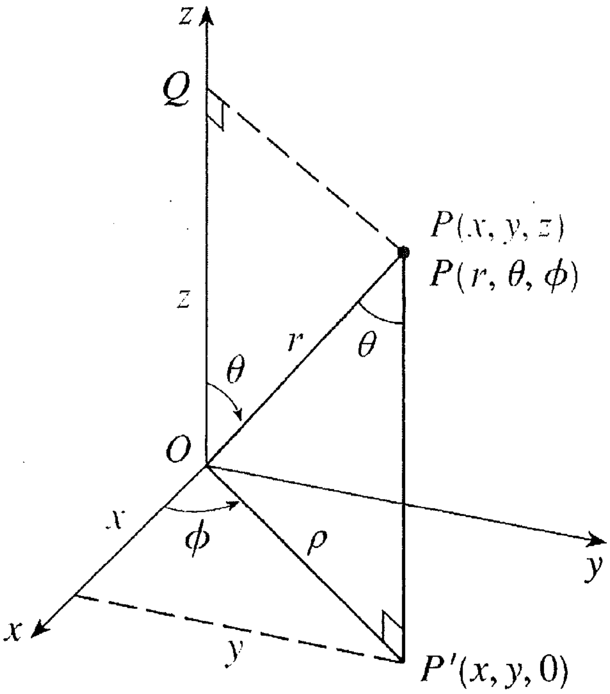
一、單選題（12題，每題5分、不倒扣）

1. Calculate the divergence of the vector field,

$$\vec{v} = \hat{r} r^3 \cos\theta + \hat{\theta} r\theta + \hat{\phi} 2\sin\phi\cos\theta$$

at the point $r = 2$, $\theta = \pi/2$, and $\phi = \pi/6$ in spherical coordinates.

(A) -2　　(B) -1　　(C) 0　　(D) 1　　(E) 2.



2. What is the angle ψ between the surfaces defined by $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at the point $(1, 2, 2)$? Note that the angle between two surfaces is the angle between their normal vectors at the point of interest.

(A) $0 \le \psi < \frac{\pi}{6}$

(B) $\frac{\pi}{6} \le \psi < \frac{\pi}{4}$

(C) $\frac{\pi}{4} \le \psi < \frac{\pi}{3}$

(D) $\frac{\pi}{3} \le \psi < \frac{\pi}{2}$

(E) $\psi = \frac{\pi}{2}$.

注意：背面有試題

3. Let . $\mathbb{M}=\begin{bmatrix} 2i & -1+i \\ 1+i & i \end{bmatrix}$ and $\mathbb{I}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(A) $\mathbb{M}$ is skew-Hermitian.

(B) The eigenvalues of any skew-Hermitian matrix must be either zero or imaginary.

(C) $\mathbb{M}$ is normal.

(D) $\mathbb{V}=(\mathbb{I}+\mathbb{M})(\mathbb{I}-\mathbb{M})^{-1}$ is unitary.

(E) All of the above are true.

4. What is the rank of the matrix $\mathbb{A}=\begin{bmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{bmatrix}$?

(A) 4　(B) 3　(C) 2　(D) 1　(E) 0.

5. Formally, the sawtooth function

$$f(x)=x+\pi \quad \text{if} \quad -\pi<x<\pi \quad \text{and} \quad f(x+2\pi)=f(x)$$

can be represented by a Fourier series as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos(nx)+b_n\sin(nx)\right],$$

where a_0, a_n, and b_n $(n\in\mathbb{N})$ are Fourier coefficients given by Euler formulas.

(A) Because $f(x)$ is not an odd function with respect to the origin, some of the $\cos(nx)$ basis terms must be included in the Fourier series.

(B) The constant term a_0 is just the average value of $f(x)$ in one period, *i.e.*, $a_0=\pi$.

(C) $b_n=-\cos(n\pi)/n$.

(D) Substituting $x=\pi/2$ into the Fourier series of this sawtooth function, we can obtain Leibniz series for π, *i.e.*, $\frac{\pi}{4}=\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n}$.

(E) None of the above is true.

6. The Fourier transform of $f(x)$ is defined by $\mathcal{F}(f)(k)=\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$.

So, which of the inverse Fourier transforms, cosine or sine, gives the result:

$$f(x)=\begin{cases}2, & |x|<1/2 \\ 0, & |x|>1/2\end{cases} \ ?$$

(A) $\displaystyle\int_0^{\infty}\frac{\sin(k/2)}{k/2}\cos(kx)\,dk$

(B) $\displaystyle\frac{2}{\pi}\int_0^{\infty}\frac{\sin(k/2)}{k/2}\cos(kx)\,dk$

(C) $\displaystyle\frac{2}{\pi}\int_0^{\infty}\frac{\sin(k/2)}{k/2}\sin(kx)\,dk$

(D) $\displaystyle\int_0^{\infty}\frac{\sin(k/2)}{k/2}\sin(kx)\,dk$

(E) None of the above is true.

7. Which set of the following integrations is correct?

(A) $\displaystyle\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx=\pi^2$ and $\displaystyle\int_{-\infty}^{\infty}\frac{\sin^4 x}{x^4}dx=\pi^4$.

(B) $\displaystyle\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx=\pi$ and $\displaystyle\int_{-\infty}^{\infty}\frac{\sin^4 x}{x^4}dx=\pi$.

(C) $\displaystyle\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx=\pi$ and $\displaystyle\int_{-\infty}^{\infty}\frac{\sin^4 x}{x^4}dx=\frac{2}{3}\pi$.

(D) $\displaystyle\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx=\pi$ and $\displaystyle\int_{-\infty}^{\infty}\frac{\sin^4 x}{x^4}dx=\frac{1}{3}\pi$.

(E) None of the above is true.

8. While the solution of the integral equation $y(t)-\int_0^t \sin(t-\tau)y(\tau)\,d\tau=t$ is $y(t)=t+\dfrac{t^3}{6}$, what is the solution of $y(t)+\int_0^t (t-\tau)y(\tau)\,d\tau=\dfrac{t^3}{6}$?

(A) $y(t)=\sin(t+t^3/6)$

(B) $y(t)=\sin(t)+t^3/6$

(C) $y(t)=\sin(t)+t$

(D) $y(t)=t-\sin(t)$

(E) None of the above is true.

注意：背面有試題

9. $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{\infty}\dfrac{1}{n^k(k+2)}\dbinom{n}{k}=?$

Hint: The approximation $\lim\limits_{n\gg 1} n! \approx \sqrt{2\pi n}\, n^n e^{-n}$ might be useful.

(A) $e+1$　(B) e　(C) $e-1$　(D) 1　(E) 0.

10. Consider a second-order ODE: $y''(z)+\dfrac{3z-1}{z(z-1)}y'(z)+\dfrac{1}{z(z-1)}y(z)=0$.

(A) The indicial equation of the given equation has a repeated root.

(B) Because $z=0$ is a regular singular point, at least one solution, say $y_1(z)$, is in the form of a Frobenius series.

(C) Let the repeated root be $\sigma_1=\sigma_2=\sigma$, then $y_1(z)$ is in a form as $y_1(z,\sigma)=\sum\limits_{n=0}^{\infty}a_n(\sigma)z^{\sigma+n}$.

(D) Accordingly, $y_2(z)=\left[\dfrac{\partial}{\partial\sigma}y_1(z,\sigma)\right]\Bigg|_{\sigma=\sigma_1}$.

(E) All of the above are true.

11. Consider a set of functions, $\{f(x)\}$, of the real variable x, defined in the interval $-\infty<x<\infty$, that $\to 0$ at least as quickly as x^{-1} as $x\to\pm\infty$. Which of the following linear operators is ***Hermitian*** when acting upon $\{f(x)\}$?

(A) $\dfrac{1}{i}\dfrac{d}{dx}+x^2$　(B) $\dfrac{d}{dx}+ix^2$　(C) $ix\dfrac{d}{dx}$　(D) $\dfrac{d}{dx}$　(E) ix.

12. The spherical harmonic functions $Y_\ell^m(\theta,\phi)$ are eigen-functions of the ***parity*** operator $\hat{\mathcal{P}}$; that is, $\hat{\mathcal{P}}Y_\ell^m(\theta,\phi)=pY_\ell^m(\theta,\phi)$, where p is the corresponding eigenvalue. According to

$$Y_\ell^m(\theta,\phi)=(-1)^m\left[\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2}P_\ell^m(\cos\theta)\exp(im\phi)$$

and $P_\ell^{-m}(x)=(-1)^m\dfrac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x)$, what the eigenvalue p should be?

(A) 1　(B) -1　(C) $(-1)^\ell$　(D) $(-1)^m$　(E) $(-1)^{\ell+m}$.

注意：背面有試題

二、計算題（4 題，每題 10 分；若該題區分 (a)、(b) 兩小題，則各 5 分；應詳列計算過程，無計算過程者不予計分）

（一）Following are the differential equations, for $t \geq 0$,

$$\ddot{x} + 2x + y = \cos 2t,$$
$$\ddot{y} + 2x + 3y = 2\cos 2t,$$

which describe a coupled system that starts from rest at the equilibrium position. (a) Show that the subsequent motion takes place along a straight line in the xy-plane. And (b) explain the resonant behavior in the solution.

Hint: The question of (a) is readily obtained by using Laplace transforms. To seek a particular solution in (b), it would be easier to take a pair of trial solutions like $x(t) = X\sin(\omega t)$ and $y(t) = Y\sin(\omega t)$.

（二）The function $y(x) = e^x$ can be spanned by all the linear functions defined in [0, 1] on the linear space.

(a) Apply the *Gram-Schmidt orthogonalization* procedure and the *inner product* of two continuous functions in $[0,1]$ to minimize the integral

$$I = \int_0^1 \left[(\alpha + \beta x) - e^x\right]^2 dx.$$

What values α and β should be?

(b) Instead of the Gram-Schmidt method, we can linearly transform the function $y(x)$ defined in $[0,1]$ to the other function $y(\xi)$ but defined in $-1 \leq \xi \leq 1$. Thus, $y(\xi)$ has the representation in terms of the Legendre orthogonal basis $\{P_n(\xi)\}$ such that

$$y(\xi) = \sum_{n=0}^{\infty} c_n P_n(\xi).$$

Converting back to $x \in [0,1]$, $y(x) = \sum_{n=0}^{\infty} a_n \hat{e}_n(x)$ where $\hat{e}_0(x) = 1$ since $P_0(\xi) = 1$, $\hat{e}_1(x) = \sqrt{3}(2x-1)$ since $P_1(\xi) = \xi$, and so on. As a result, what is the *normalized* basis $\hat{e}_3(x)$?

Hint: $(n+1)P_{n+1}(\xi) = (2n+1)\xi P_n(\xi) - nP_{n-1}(\xi)$.

（三）Solve $3\dfrac{\partial u(x,y)}{\partial x} - 2\dfrac{\partial u(x,y)}{\partial y} + u(x,y) = x$ with $u = 0$ on $2y - 3x = 0$.

（四）Compute the complex integral: $\oint_{C:\,|z-i|=1} \csc\left(\frac{1}{z-i}\right) dz$.

Hint: Consider the pole at infinity.