1. Consider a pollutant concentration field $\mathbf{F}(x,y,z)$ (a vector field representing mass flux in the atmosphere) over a 3D region. We are interested in how pollutants accumulate or disperse within a given control volume.

   Part 1: Divergence and Conservation of Mass (9%):
   - **(a)** Define the divergence $\nabla \cdot \mathbf{F}$ and discuss its physical interpretation in atmospheric pollution modeling.
   - **(b)** Explain how the divergence theorem (Gauss's theorem) links the net flux of pollutants through a closed surface to the accumulation of pollutants inside the volume.

   Part 2: Example Calculation (8%):
   - Suppose $\mathbf{F}(x,y,z) = (xy, x^2z, yz^2)$. Compute $\nabla \cdot \mathbf{F}$.
   - Interpret the sign and magnitude of $\nabla \cdot \mathbf{F}$ at a given point (x,y,z). Where might pollutants tend to accumulate based on positive or negative divergence?

   Part 3: Practical Application (8%):
   - Describe how meteorological data (e.g., wind velocity fields) and pollutant emission data could be combined into $\mathbf{F}$ to predict pollution "hot spots."
   - Briefly discuss how you might extend this model to include chemical reactions or transformations of pollutants in the atmosphere.

2. A sudden chemical spill in a forest results in the dispersion of a pollutant into the surrounding soil and air. The pollutant's concentration, $C(t)$, at a monitoring site is influenced by natural attenuation processes and environmental damping forces such as vegetation absorption and chemical degradation.

The pollutant dynamics are modeled by the second-order differential equation:

$$m\frac{d^2C(t)}{dt^2} + c\frac{dC(t)}{dt} + k\,C(t) = F(t),$$

where:

- m represents the "inertial" effect of pollutant dispersion across the ecosystem,
- c represents damping forces such as vegetation uptake and microbial degradation,
- k represents a restoring force that models the natural re-equilibration of pollutant levels,
- $F(t)$ represents the source term, modeling the ongoing input of the pollutant into the environment (e.g., leaching from the spill site).

Part 1: Solving the Differential Equation (13%)

1. Take the Laplace transform of the equation (assuming the initial pollutant concentration $C(0) = 0$ and initial rate of change $\frac{dC}{dt}(0) = 0$
2. Solve for $C(s)$, the Laplace transform of the pollutant concentration.
3. Assume $F(t) = Ae^{-\alpha t}$, where A is the initial spill intensity and $\alpha > 0$ models the decay of spill intensity over time. Use the Laplace transform of $F(t)$ to find $C(s)$.
4. Apply the inverse Laplace transform to determine $C(t)$. Analyze the concentration behavior under different damping coefficients (c).

Part 2: Taylor Expansion for Rapid Assessment (12%)

1. In emergency response scenarios, $F(t)$ can be approximated using a Taylor series expansion around $t = 0$. Expand $F(t) = Ae^{-\alpha t}$ into a Taylor series up to the t^3 term.
2. Substitute the Taylor-expanded $F(t)$ into the original differential equation. Solve the approximate differential equation using the Laplace transform.
3. Compare the approximate solution with the exact solution derived in Part 1. Discuss when the Taylor approximation is valid.
4. Analyze how the pollutant concentration changes over time for a rapidly decaying spill (α large) versus a slowly decaying spill (α small).

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3. Solve the following equations: **(25%)**

(a) $(x^2+4)y' = 3x - 2xy$

(b) $y' = \frac{x+2y}{y+2x}$

(c) $xy' = \frac{1}{2}y^2 + y$

(d) $x^3y' = x^2y - 3y^3$

(e) $y'' - 2y' + y = x^2e^x$

4. Find the general solution of the initial value problem: **(15%)**

(a) $y' - 3y = -12\,y^2,\ y(0) = 2$

(b) $y'\cos^2 x + 3y = 1,\ y\left(\frac{1}{4}\pi\right) = \frac{4}{3}$

5. Solve $y'' + 4\,y' + 4\,y = (3+x)e^{-2x},\ y(0) = 0,\ y'(0) = 7$ **(10%)**

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