考試科目	線性代數	系所別	應用數學系	考試時間	2月12日(星期三) 第四節

Notation.

$P_n(\mathbb{R})$: the set of polynomials of degree at most n.
I : the identity matrix.
$\mathbb{R}^{n\times m}$: the set of $n \times m$ real matrices.

1 Let $T : V \to W$ be a linear transformation. Determine "true" or "false" for the following statements. If "true", prove it; if "false", give a counterexample.

a. (10%) If $\{v_1, v_2\}$ is linearly independent in V, then $\{T(v_1), T(v_2)\}$ is linearly independent in W.

b. (10%) If $\{T(v_1), T(v_2)\}$ is linearly independent in W, then $\{v_1, v_2\}$ is linearly independent in V.

2 (10%) Suppose that W_1 and W_2 are subspaces of a vector space V. Show that $W_1 + W_2 = \{x + y \mid x \in W_1,\ y \in W_2\}$ is a subspace of V.

3 Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(p(x)) = p(x-2) + (x+1)p'(x)$, where $p'(x)$ is the first derivative of $p(x)$.

a. (5%) Show that T is a linear transformation.

b. (5%) Let $\beta = \{x^2, x, 1\}$ be the ordered basis for $P_2(\mathbb{R})$. Find $[T]_\beta^\beta$.

c. (5%) Find the eigenvalues and eigenvectors of T.

4 (10%) Let $A \in \mathbb{R}^{n\times n}$ be invertible and $x,\ y \in \mathbb{R}^n$, $c \in \mathbb{R}$. Define

$$M = \begin{bmatrix} A & x \\ y^T & c \end{bmatrix}.$$

Show that M is invertible if and only if $c \neq y^T A^{-1} x$.

5 Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ -8 \\ 12 \\ 2 \end{bmatrix}$.

a. (10%) Find the least squares solution of the linear system $Ax = b$.

b. (5%) Find the orthogonal projection of b on to the column space of A.

6 Let

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix}.$$

a. (10%) Find an invertible matrix Q such that $Q^{-1}AQ$ is a diagonal matrix.

b. (5%) Find the eigenvalues and eigenvectors of $A^3 - 3A^2 + 4I$.

c. (5%) Describe the set $W = \{p(x) | p(x)$ is a polynomial and $p(A) = 0\}$.

7 (10%) Let $A \in \mathbb{R}^{n\times m}$ with $\text{rank}(A) = m$. Show that $A^T A$ is invertible.

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。