國立臺灣師範大學 113 學年度碩士班招生考試試題

科目：**數學基礎** 適用系所：**資訊工程學系**

注意：1.本試題共 3 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Some notations:

- A *vector* refers to a column vector with real entries, for example, $\boldsymbol{v} = \begin{bmatrix}1\\2\\3\end{bmatrix} \in \mathcal{R}^3$.
- A linear transformation where the domain and codomain equal $\mathcal{R}^n$ is called a **linear operator** on $\mathcal{R}^n$.

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1. (14 points)Suppose that the reduced row echelon form R and three columns of $A = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \cdots \quad \boldsymbol{a}_5]$, are given by $R = \begin{bmatrix}1 & 2 & 0 & 0 & -2\\0 & 0 & 1 & 0 & 3\\0 & 0 & 0 & 1 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix}6\\10\\4\\2\end{bmatrix}$, $\boldsymbol{a}_3 = \begin{bmatrix}2\\0\\-1\\-1\end{bmatrix}$, and $\boldsymbol{a}_4 = \begin{bmatrix}2\\-1\\3\\2\end{bmatrix}$.

   (a) (6 points) Determine the matrix A.

   (b) (2 points) Determine the dimension of the column space of A, i.e. Col A.

   (c) (2 points) Determine the dimension of the null space of A, i.e. Null A.

   (d) (4 points) Determine a vector form for the general solution of the system of linear equations $A\boldsymbol{x} = \boldsymbol{0}$, where $\boldsymbol{x} = \begin{bmatrix}x_1\\x_2\\\vdots\\x_5\end{bmatrix}$.

2. (a) (5 points) Let A be a 3×3 matrix such that $\det A = -9$, the determinant of A. The (*i,j*) entry of A is denoted by a_{ij}. Evaluate the determinant of matrix

$$\begin{bmatrix}a_{11} - 3a_{21} & a_{12} - 3a_{22} & a_{13} - 3a_{23}\\4a_{21} & 4a_{22} & 4a_{23}\\2a_{31} - 5a_{21} & 2a_{32} - 5a_{22} & 2a_{33} - 5a_{23}\end{bmatrix};$$

(b) (5 points) T is a linear operator on $\mathcal{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) =$ $\begin{bmatrix} -4x_1 - 2x_2 \\ cx_2 \\ 4x_1 + 4x_2 - 2x_3 \end{bmatrix}$ for some scalar c. Determine all the values of c for which T is not diagonalizable.

3. (a) (7 points) Find an explicit formula for the **reflection** operator T_W of $\mathcal{R}^3$ about the plane W, where $W = \left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathcal{R}^3 : x_1 - x_2 + x_3 = 0\right\}$.

   (b) (6 points) Given the plane W in (a) and a vector $\boldsymbol{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ in $\mathcal{R}^3$, find the unique vector $\boldsymbol{w}$ in W that is closest to $\boldsymbol{u}$.

4. (13 points) Let $A = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$, a symmetric matrix.

   (a) (7 points) Find an orthonormal basis of eigenvectors and their corresponding eigenvalues of A.

   (b) (6 points) Use this information to obtain a spectral decomposition of A^4.

5. (15 points) Let G be an undirected simple graph with vertex set V. We say that a vertex subset U is "good" if every vertex in U has exactly one neighbor in V-U. We say that a vertex subset U is "not bad" if every vertex in V-U has exactly one neighbor in U.

   (a) (5 points) Prove or disprove: If a vertex subset is "good", then it is "not bad".

   (b) (10 points) Prove or disprove: If a vertex subset is "not bad", then it is "good".

6. (25 points) Let M be an m-by-n matrix. Each entry is set to be 0 or 1 with probability 1/2, independently. Let Ω be the sample space consisting of all outcomes of M. For example, below is a possible outcome for a 5-by-4 matrix.

	1	2	3	4
1	0	0	0	0
2	1	0	1	0
3	0	1	1	1
4	1	0	1	0
5	0	1	0	1

(a) (5 points) An entry $M_{i,j}$ is "feasible" if $M_{i-1,j}$, $M_{i,j+1}$, $M_{i+1,j}$, $M_{i,j-1}$ are all set to 1. Let $X(i, j)$ be the event that entry $M_{i,j}$ is feasible. In the example above, only entries at (3, 3) and (4, 2) are feasible. Prove or disprove: For $(a, b) \neq (c, d)$, the events $X(a, b)$ and $X(c, d)$ are independent.

(b) (5 points) Let W: $\Omega \rightarrow \mathbb{R}$, a random variable, be the number of feasible entries in M. Please compute E[W] (the expected value of W).

(c) (5 points) Alice observes that for $m \geq 3$ and $n \geq 2^m+1$ there are always four distinct entries $M_{i,j}$, $M_{i,j'}$, $M_{i',j}$, $M_{i',j'}$ that are all 0 or all 1. For example, in the matrix above, let $(i, j, i', j') = (1, 1, 5, 3)$. Entries at the corresponding positions are all 0. Prove that Alice's observation is guaranteed to hold, or give a counterexample.

(d) (10 points) Bob observes that for $m \geq 3$ and $n \geq 7$ there are always four distinct entries $M_{i,j}$, $M_{i,j'}$, $M_{i',j}$, $M_{i',j'}$ that are all 0 or all 1. Prove that Bob's observation is guaranteed to hold, or give a counterexample.

7. (10 points) Find the SECOND SMALLEST positive integer that is a solution to the following system of congruences.

$$x \equiv 20 \pmod{26}$$
$$x \equiv 8 \pmod{11}$$
$$x \equiv 18 \pmod{57}$$