國立臺灣師範大學 113 學年度碩士班招生考試試題

科目：數值分析 適用系所：**數學系**

注意：1.本試題共 1 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

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1. Let f be a function of a real variable defined as $f(x) = x + \sin(x) - 1$ and let

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

   (a) (5 points) Describe the largest domain in which the real-valued function g is well-defined.

   (b) (10 points) Show that g has a unique fixed point in $(0, \frac{\pi}{2})$.

   (c) (10 points) Show that the fixed point of g is a root of the equation $x + \sin(x) = 1$.

   (d) (5 points) Show that g'' is continuous on $(0, \frac{\pi}{2})$.

   (e) (5 points) Show that $|g''(x)|$ is bounded on $(0, \frac{\pi}{2})$.

   (f) (15 points) Let $x_* \in (0, \frac{\pi}{2})$ be the fixed point of g and let $x_0 \in (0, \frac{\pi}{2})$. Show that the sequence $\{x_k\}_{k=0}^{\infty}$ defined by the iteration $x_{n+1} = g(x_n)$ converges at least quadratically to x_*.

2. Given

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

   We define a linear operator T on $\mathbb{R}^3$ as

$$T(\mathbf{x}) = B\mathbf{x} + \mathbf{f},$$

   where $B = I - D^{-1}A$, $\mathbf{f} = D^{-1}\mathbf{b}$,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

   (a) (10 points) Show that T has a unique fixed point.

   (b) (10 points) Show that A is strictly diagonally dominant.

   (c) (10 points) Show that every eigenvalue λ of the matrix B satisfies $|\lambda| < 1$.

   (d) (20 points) Let $\mathbf{x}_0 \in \mathbb{R}^3$ be an arbitrary initial vector. Show that the sequence $\{\mathbf{x}_k\}_{k=0}^{\infty}$ defined by the iteration $\mathbf{x}_{k+1} = T(\mathbf{x}_k)$ converges to the solution of the linear system $A\mathbf{x} = \mathbf{b}$.