國立臺灣師範大學 113 學年度碩士班招生考試試題

科目：代數　　　　　　　　　　　　　　　　適用系所：數學系

注意：1.本試題共 1 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

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1. (10 points) Let S_n be the symmetric group of degree n. Let C_m be the cyclic group of order m. Is the group $S_3 \times C_4$ isomorphic to S_4? Justify your answer.

2. (10 points) Let G, H be groups and let $\theta : G \to H$ be a surjective group homomorphism. If N is a normal subgroup of G and G/N is abelian, prove that $\theta(N)$ is normal in H and $H/\theta(N)$ is abelian.

3. For convenience, we say that a finite group G is CLT if G has the following property: "For any positive integer d such that d divides $|G|$, G has a subgroup of order d."
   (a) (10 points) Let G be a group of order 20. Show that G is CLT.
   (b) (10 points) Let p be a prime. If G is an abelian group of order p^k for some positive integer k, is it always true that G is CLT? Justify your answer.

4. Let $\mathbb{Z}$ be the ring of integers. Let $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers where $i^2 = -1$. Let $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.
   Prove or disprove each of the following statements.
   (a) (10 points) The ideal $(x + 1)$ is a prime ideal in the polynomial ring $\mathbb{Z}[x]$.
   (b) (10 points) 23 is irreducible in $\mathbb{Z}[i]$.
   (c) (10 points) The ideal $I = (2, 1 + \sqrt{-5})$ is a principal ideal in $\mathbb{Z}[\sqrt{-5}]$.
   (d) (10 points) $\mathbb{Z}[i]/(5 + i)$ is a field.

5. Let $\mathbb{Q}$ be the field of rational numbers. Let $f(x) = x^{35} - 15x^2 + 45$ and $g(x) = x^3 - 5x + 3$ be polynomials over $\mathbb{Z}$. Suppose that α is a complex number and $f(\alpha) = 0$. Let $\beta = g(\alpha)$.
   (a) (10 points) Determine $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.
   (b) (10 points) Determine the degree of the minimal polynomial of β over $\mathbb{Q}$.