國立臺灣師範大學 113 學年度碩士班招生考試試題

科目：**高等微積分** 適用系所：**數學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

1. (10 points) Show the following identity:

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

2. (5+5 points) Evaluate the following integrals: (You only need to write down the answers.)

(1) $\int_{-2}^{2} xd(|x|)$. (2) $\int_0^\pi \cos x d(\sin x)$.

3. (10 points) Let $p > 0$. Discuss the convergence or divergence for the series

$$\sum_{n=3}^{\infty} \frac{1}{n(\log n)(\log\log n)^p}$$

Justify your result.

4. (5+5 points) Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \frac{1}{n}e^{-n^2x^2}, \ x \in \mathbb{R},\ n \in \mathbb{N}.$$

(1) Prove that $f_n \to 0$ uniformly on $\mathbb{R}$.

(2) Prove that $f_n' \to 0$ pointwisely on $\mathbb{R}$ but not uniformly.

5. (5+10 points) Evaluate the extreme values of the following functions: (You only need to write down the answers.)

(1) $f(x, y, z) = x - y + z$ subject to the constraint $x^2 + y^2 + z^2 = 2023$.

(2) $g(x, y, z) = x^3 + y^3 + z^3$ subject to the constraints $x^2 + y^2 + z^2 = 1$ and $x + y + z = 1$.

6. (15 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that $D_{12}f$ and $D_{21}f$ are <u>not</u> continuous at $(0,0)$.

7. (5+5 points) Let $f:(0,1)\to\mathbb{R}$ be a differentiable function where $a,b\in\mathbb{R}$.

   (1) Prove or disprove that $|f'(x)|$ is bounded for all $x\in(0,1)$?

   (2) If f have a bounded derivative, that is, $|f'(x)|<\infty$ for all $x\in(0,1)$. Show that f is uniformly continuous on $(0,1)$.

8. (5+5 points) Let $f:\mathbb{R}\to\mathbb{R}$ be a function.

   (1) Supoose that f is continuous on $[0,1]$. Show that f is Riemann integrable on $[0,1]$, that is, $\int_0^1 f(x)dx$ exists.

   (2) Define

   $$f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q},\\ 0 & \text{if } x\in\mathbb{Q}^c.\end{cases}$$

   Show that f is not Riemann integrable on $[0,1]$.

9. (5+5 points) Let $f:\mathbb{R}^3\to\mathbb{R}$ be defined by

   $$f(x,y,z)=x^3+y^3+z^3-3xyz-1.$$

   (1) Show that there exists a function $\phi:\mathbb{R}^2\to\mathbb{R}$ such that

   $$f(x,y,\phi(x,y))=0$$

   in a neighborhood of $(0,0,1)$.

   (2) Find the <u>total differentiate</u> of $\phi(x,y)$.