國立中央大學 113 學年度碩士班考試入學試題

所別： 數學系 碩士班 數學組(一般生)
數學系 碩士班 應用數學組(一般生)
數學系 碩士班 應用數學組(在職生)

科目： 線性代數

＊本科考試禁用計算器

Please justify your answer properly to get full credits.

1. Prove the following statements:
   (a) (5%) If A and B are $m \times n$ matrices, then $\text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B)$.
   (b) (5%) If $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times p}$ for some field $\mathbb{F}$, then $\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$.
   (c) (5%) If $A \in \mathbb{R}^{m\times n}$, then $\text{rank}(A^\top A) = \text{rank}(A)$.

2. Let

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

   (a) (5%) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.
   (b) (5%) Find A^n, where $n \in \mathbb{N}$.
   (c) (5%) Find $\exp(A)$.

3. (10%) Let $A \in \mathbb{R}^{m\times n}$ with $m > n$ and $\mathbf{b} \in \mathbb{R}^{m\times 1}$. Prove that $\mathbf{x}$ minimizes the least squares error $\|\mathbf{b} - A\mathbf{x}\|_2^2$ *if and only if* $\mathbf{x}$ solves the normal equation $A^\top A\mathbf{x} = A^\top \mathbf{b}$.

4. Let $\mathbf{a} \in \mathbb{R}^{n\times 1}$ and $A = \mathbf{a}\mathbf{a}^\top \in \mathbb{R}^{n\times n}$.
   (a) (5%) Given $n > 1$, prove or disprove that A is singular.
   (b) (20%) If $\mathbf{a} = [1,\ 2,\ 3,\ 4]^\top$, find all eigenvalues of A.

5. Let

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 3 & 5 \end{bmatrix}.$$

   (a) (10%) Find an orthonormal basis for the vector space spanned by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.
   (b) (5%) Find the QR-decomposition of A.

6. (10%) Assume $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^{m\times 1}$. Prove that $A\mathbf{x} = \mathbf{b}$ has a solution *if and only if* given $\mathbf{y}$ with $A^\top \mathbf{y} = \mathbf{0}$ implies $\mathbf{b}^\top \mathbf{y} = 0$.

7. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Prove the following statements:
   (a) (5%) All eigenvalues of A are real.
   (b) (5%) Given two distinct eigenvalues of A, λ_1 and λ_2, corresponding to the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then $\mathbf{v}_1 \perp \mathbf{v}_2$.