國立臺灣師範大學 112 學年度碩士班招生考試試題

科目：**代數** 適用系所：**數學系**

注意：1.本試題共 1 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

- Let $\mathbb{Z}$ be the ring of integers. Let $\mathbb{Q}$ be the field of rational numbers.

1. Let p be a prime and let G be a cyclic group of order p^3.
   (a) (5 points) Find the number of elements of order p^2 in G.
   (b) (10 points) Find the number of elements of order p in the direct product $G \times G$.

2. Let N, A, B be subgroups of the group G. Suppose that $N \lhd G$ and $A \lhd B$.
   (a) (5 points) Show that $AN \lhd BN$.
   (b) (10 points) If B/A is abelian, show that BN/AN is abelian.

3. Let G be the dihedral group of order 144. So G is a group generated by two elements a and b where a has order 72, b has order 2 and $bab^{-1} = a^{-1}$.
   (a) (10 points) Determine the number of Sylow 2-subgroups of G.
   (b) (10 points) Find the center of G.

4. Prove or disprove each of the following statements.
   (a) (10 points) In an integral domain, every prime element is irreducible.
   (b) (10 points) $\mathbb{Z}[\sqrt{-7}]$ is a unique factorization doamin (UFD).
   (c) (10 points) If M is a maximal ideal of an integral domain R, then $M[x]$ is a maximal ideal of the polynomil ring $R[x]$.

5. (10 points) Let α be the real number $7+\sqrt[5]{1+\sqrt{3}}$. Find the degree of the field extension $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

6. (10 points) Let $\alpha = \sqrt[3]{2}$ and $\beta = \sqrt[3]{5}$. Is it true that $\mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\alpha + \beta)$? Justify your answer.