國立臺灣師範大學 112 學年度碩士班招生考試試題

科目：高等微積分 適用系所：**數學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

1. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples for the false ones.

   (a) (6 points) Let f be a differentiable function on $\mathbb{R}$. If $f(0) = 1$ and $|f'(x)| \leq M$ for all $x \in \mathbb{R}$, then $|f(x)| \leq M|x| + 1$.

   (b) (6 points) If f and g are differentiable on $[a, b]$, and if f' and g' are Riemann integrable on $[a, b]$, then

   $$\int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx = 0$$

   if and only if $f(a)g(a) = f(b)g(b)$.

   (c) (6 points) Every continuous real function $f(x)$ defined on an closed interval $[a, b]$ has an antiderivative.

   (d) (6 points) If $f \times f$ is Riemann integrable on $[a, b]$, then f is Riemann integrable on $[a, b]$.

   (e) (6 points) A continuous function maps Cauchy sequences to Cauchy sequences.

2. (10 points) Suppose that $f : (-\infty, \infty) \to \mathbf{R}$ is continuous and that

   $$\lim_{x \to \pm\infty} f(x) = L.$$

   Prove that the function f is uniformly continuous on $\mathbf{R}$.

3. (a) (5 points) If f is a bounded function. Prove that

   $$\sup_{x \in [a,b]} f \times f = (\sup_{x \in [a,b]} |f|)^2.$$

   (b) (5 points) If f is integrable on $[a, b]$, prove that $f \times f$ is integrable on $[a, b]$. The following formula can be used directly.

   $$\inf_{x \in [a,b]} f \times f = (\inf_{x \in [a,b]} |f|)^2.$$

4. Let f be a continuous function on $[a, b]$.

   (a) (5 points) Describe the definitions of the lower sum and the lower integral of f.

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(b) (5 points) Let $f(x_0) \neq 0$ for some $x_0 \in [a,b]$. Prove that

$$(L)\int_a^b |f(x)|dx > 0,$$

where the symbol $(L)\int_a^b$ is the lower integral.

(c) (3 points) By the result (b), show that $\int_a^b |f(x)|dx = 0$ if and only if $f(x) = 0$ for all $x \in [a,b]$.

5. (10 points) Prove that $f(x) = \sum_{k=1}^{\infty} \sin(kx)/k$ converges for each $x \in \mathbf{R}$.

6. (a) (3 points) Describe the Green's theorem.

(b) (12 points) Give two functions defined on $\mathbf{R}^2 \setminus (0,0)$ as follows.

$$f(x,y) = -\frac{y}{x^2+y^2}, \quad g(x,y) = \frac{x}{x^2+y^2}.$$

Let C be a simple, closed, and positively oriented curve. Compute

$$\int_C f dx + g dy.$$

7. (a) (2 points) Describe the definition of a Jordan region.

(b) (10 points) Let E be a Jordan region in $\mathbf{R}^2$, where $(x,y) \in E$. Prove that

$$\lim_{k\to\infty} \iint_E \cos(x/k) e^{y/k} dA$$

exists, and find its value.

(試題結束)