國立臺灣師範大學 112 學年度碩士班招生考試試題

科目：**數學基礎**　　　　　　　　適用系所：**資訊工程學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Some notation:

- A *vector* refers to a column vector with real entries, for example, $\boldsymbol{v} = \begin{bmatrix}1\\2\\3\end{bmatrix} \in \mathcal{R}^3$.
- The **span** of a finite nonempty subset of $\mathcal{R}^n$ is a subspace of $\mathcal{R}^n$.
- The **transpose** of matrix A is denoted by A^T whose (i,j)-entry is the (j,i)-one of A.
- Let T be a linear operator on $\mathcal{R}^n$ and $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}$ be a basis for $\mathcal{R}^n$. The matrix $[[T(\boldsymbol{b}_1)]_{\mathcal{B}} \quad [T(\boldsymbol{b}_2)]_{\mathcal{B}} \quad \cdots \quad [T(\boldsymbol{b}_n)]_{\mathcal{B}}]$ is called the **matrix representation of T with respect to** $\mathcal{B}$ or the $\mathcal{B}$**-matrix of T**, denoted by $[T]_{\mathcal{B}}$.

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1. (a) (4 points) Determine the value of r for which $\boldsymbol{v}$ is in the span of S, where

$$S = \left\{\begin{bmatrix}-1\\2\\2\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}\right\} \text{ and } \boldsymbol{v} = \begin{bmatrix}2\\r\\-8\end{bmatrix}.$$

(b) (6 points) Find a basis for the column space and null space of

$$\begin{bmatrix}-1 & 2 & 1 & -1\\ 2 & -4 & -3 & 0\\ 1 & -2 & 0 & 3\end{bmatrix}.$$

2. (10 points) Suppose that $T: \mathcal{R}^3 \to \mathcal{R}^3$ is a linear transformation such that

$$T\left(\begin{bmatrix}-1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix},\ T\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right) = \begin{bmatrix}4\\-1\\2\end{bmatrix} \text{ and } T\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right) = \begin{bmatrix}-2\\3\\-2\end{bmatrix}.$$

Determine $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)$ for any $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ in $\mathcal{R}^3$.

3. (10 points) Let $\mathcal{A} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a basis for $\mathcal{R}^n$. Then, $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_1+\boldsymbol{u}_2, \ldots, \boldsymbol{u}_1+\boldsymbol{u}_n\}$ is also a basis for $\mathcal{R}^n$. If $\boldsymbol{v}$ is a vector in $\mathcal{R}^n$ and $[\boldsymbol{v}]_{\mathcal{A}} = \begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}$, compute $[\boldsymbol{v}]_{\mathcal{B}}$.

4. (10 points) For the matrix $A = \begin{bmatrix}2 & 4 & 4\\ 4 & 17 & -1\\ 4 & -1 & 17\end{bmatrix}$, find an orthogonal matrix P and a diagonal matrix D such that $P^TAP = D$.

5. (10 points) Let W, a subspace, be the orthogonal complement of Span $\left\{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}\right\}$.

(a) Find the orthogonal projection matrix P_W for W. (6 points)

(b) Find the vector $\boldsymbol{w}$ in W that is closest to vector $\boldsymbol{v} = \begin{bmatrix}2\\-1\\1\\2\end{bmatrix}$. (4 points)

6. (15 points) Let A be a set of size m, and let B be a set of size n, where m and n are positive integers.
(a) How many functions $f: A \to B$ can be defined?
(b) How many 1-1 functions $f: A \to B$ can be defined?
(c) How many onto functions $f: A \to B$ can be defined for $n = 2$?

7. (5 points) Let p be a prime number, and let x be a positive integer.
Show that if $1 \cdot 2 \cdot \ldots \cdot (p-1) = 1 \cdot 2 \cdot \ldots \cdot (p-1) \cdot x$, then $x \equiv 1 \pmod{p}$.

8. (10 points) Let $A = \{a, b\}$. Find all possible binary relations R on A such that R is **neither** reflexive **nor** transitive, or show that there is no such binary relation.

9. (15 points) Let G be a **connected simple** graph with n vertices and m edges, where $n \geq 2$. Assume that there are n-1 vertices whose degrees are pairwise distinct.
(a) Show that exactly two vertices have degree $\lfloor n/2 \rfloor$.
(b) Given $m + n = 155$, find n and m.

10. (5 points) Let n be a positive integer. Derive a closed formula for

$$\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k}.$$

Note that the operations that can be used in a closed formula are arithmetic operations (+, −, ×, /) and taking powers.