科目：線性代數　　　　系所組：數學系甲組

1. (10%) For scalars b and c, determine a condition for which the system of linear equations

$$\begin{aligned} x + y \qquad &= 1 \\ y - 2z &= b \\ x \qquad + 2z &= c \end{aligned}$$

has no solution.

2. Let

$$A = \begin{pmatrix} -2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

(a) (16%) Calculate an orthonormal basis of eigenvectors of A.

(b) (8%) Determine orthogonal matrices P and P^{-1} so that $P^{-1}AP$ is a diagonal matrix.

3. Let A be a 5×5 matrix with the characteristic polynomial

$$f(t) = -t^5 + t^4 - 2t^2 - 3.$$

(a) (8%) Determine rank(A).

(b) (12%) Calculate the determinant of the matrix $A^4 - A^3 + 2A$.

4. (30%, 6% each) Prove or disprove each of the following statements.

(a) Let W_1 and W_2 be subspaces of a vector space V. Then $W_1 \cup W_2$ is a subspace of V.

(b) Let V and W be vector spaces and $T : V \to W$ be a linear transformation. If $\dim(V) < \dim(W)$, then T cannot be onto.

(c) Let $Ax = b$ be a system of m linear equations in n unknowns. If rank$(A) = m$, then $Ax = b$ has a solution.

(d) Let I_n be the $n \times n$ identity matrix. If A is an $n \times n$ matrix such that $A^2 = A$, then $I_n - 2A$ is invertible.

(e) Let A be as in part d). If the characteristic polynomial of A splits, then A is diagonalizable.

5. (16%) Let W be a finite-dimensional subspace of an inner product space V and let $x \in V \backslash W$. Denote $W^{\perp}$ the orthogonal complement of W. Prove that there exist unique vectors $u \in W$ and $y \in W^{\perp}$ such that $x = u + y$. [Hint : Consider the orthogonal projection of x on W.]

※ 注意：1.考生須在「彌封答案卷」上作答。
2.本試題紙空白部份可當稿紙使用。
3.考生於作答時可否使用計算機、法典、字典或其他資料或工具，以簡章之規定為準。