國立清華大學 101 學年度碩士班考試入學試題

系所班組別：數學系 應用數學組
考試科目（代碼）：線性代數（0202）

　　＊請在【答案卷、卡】作答

1. [30%] True or false? With a reason.

(1) There exists a matrix A such that the vector $(1,1,1) \in \mathbb{R}^3$ is contained in the nullspace of A and the vectors $(1,2,-3)$ and $(-2,3,5)$ is contained in the row space of A.

(2) If two real n by n matrices A, B are diagonalizable (over $\mathbb{R}$), then AB is also diagonalizable.

(3) Let A be a real m by n matrix. If $A^{\mathrm{T}}A$ is positive definite, then AA^{T} is also positive definite where A^{T} denotes the transpose of A.

2. [10%] Let $T\colon \mathrm{M}_{4\times 4}(\mathbb{R}) \to \mathrm{M}_{4\times 4}(\mathbb{R})$ be the linear transformation defined by $T(A) := A + A^{\mathrm{T}}$ where $\mathrm{M}_{4\times 4}(\mathbb{R})$ denotes the vector space of real 4 by 4 matrices. Find the rank of T.

3. [10%] Let $P\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the (orthogonal) projection onto the subspace spanned by

$$(1,0,1,0), (0,1,0,1), (1,0,0,1), (1,1,0,0)$$

Find the matrix P.

4. [10%] Find the determinant of the 6 by 6 matrix

$$\begin{bmatrix} 1 & 4 & 9 & 16 & 25 & 36 \\ 4 & 9 & 16 & 25 & 36 & 49 \\ 9 & 16 & 25 & 36 & 49 & 64 \\ 16 & 25 & 36 & 49 & 64 & 81 \\ 25 & 36 & 49 & 64 & 81 & 100 \\ 36 & 49 & 64 & 81 & 100 & 121 \end{bmatrix}.$$

5. [10%] Find the decomposition $A = QR$ where

$$A = \begin{bmatrix} 2 & 0 & 8 \\ 1 & 1 & 5 \\ 0 & 0 & 0 \\ 2 & 4 & 3 \end{bmatrix},$$

Q is orthogonal (i.e., $Q^{\mathrm{T}} \cdot Q = I$), and R is upper-triangular.

6. [10%] We want to fit a plane $z = C + Dx + Ey$ (in $\mathbb{R}^3$) to the four points:

$z = 3$ at $x = 1$, $y = 1$;

$z = 5$ at $x = 2$, $y = 1$;

$z = 6$ at $x = 0$, $y = 3$;

$z = 0$ at $x = 0$, $y = 0$.

(1) Find 4 equations in 3 unknowns to pass a plane through the points (if there is such a plane).

(2) Find 3 equations in 3 unknowns for the best least-square solution.

7. [10%] The *norm* of a real n by n matrix A is the number $\|A\|$ defined by

$$\|A\| = \max_{\mathbf{x} \neq \mathbf{0},\ \mathbf{x} \in \mathbb{R}^n} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$$

where $\|\mathbf{x}\|$ denotes the Euclidean norm of $\mathbf{x}$. Prove that if A is symmetric, then $\|A\|$ is equal to the largest eigenvalue of A.

8. [10%] Use the inner product defined by

$$\langle p(t), q(t) \rangle = \int_{-1}^{1} p(t)q(t)\,\mathrm{d}t$$

on $\mathbf{P}_3(\mathbb{R})$ (the space of real polynomials of degree less than or equal to 3) and let W be the subspace spanned by $\{t, t^2\}$.

(1) Find the orthogonal complement $W^\perp$.

(2) Write the polynomial $1+t+t^2+t^3$ as $p(t)+q(t)$ with $p(t) \in W$ and $q(t) \in W^\perp$.