計算題應詳列計算過程，無計算過程者不予計分

1. Fourier transform of a function $f(x)$ is defined by

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx.$$

In what follows, you can use the formula

$$\delta(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi x} dx,$$

where $\delta(\xi)$ is the Dirac delta function.

(1)(申論題 5%) Let the Fourier transform of $xf(x)$ be $G(\xi)$, namely

$$G(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x f(x) e^{-i\xi x} dx.$$

Prove $G(\xi) = -i \frac{d}{d\xi} F(\xi)$.

(2) (計算題 5%) Obtain the Fourier transform of $f(x) = x$.

(3) (申論題 10%) Prove that the inverse Fourier transform of $F(\xi)$ is equal to $f(x)$.

2. Solve the following differential equations.

(1) (計算題 5%) $\frac{d}{dx} f(x) = -\frac{x}{f(x)}$

(2) (計算題 10%) $\frac{d}{dx} f(x) + 9f(x) = e^{-x}$.

(3) (計算題 10%) $\frac{d}{dx} f(x) = x^2 f(x)$.

[Hint: Assume $f(x) = \sum_{n=0}^{\infty} b_n x^{n+\lambda}$ $(b_0 \neq 0)$, and find b_n and λ. After that prove $f(x) = b_0 \exp(x^3/3)$.]

注意:背面有試題

3. Find the extrema (local maxima or minima) of the following functions or functionals.

(1) (計算題 5%) $I(x) = x^3 + 9x^2 + 24x.$

(2) (計算題 5%) $I(x, y, z) = xyz$, with constraints $x^2 + y^2 + z^2 = 1$ and $x, y, z > 0.$

(3) (計算題 10%) Using the Euler-Lagrange equation, find the function $f(x)$ that achieves an extremum of a functional $I[f(\cdot)] = \int_2^5 \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$, with boundary conditions $f(2) = 3$ and $f(5) = 9.$

(4) (申論題 10%) Obtain the differential equation of $f(x)$ that achieves extremum of a functional $I[f(\cdot)] = \int_a^b L\left(f, \frac{df}{dx}, \frac{d^2f}{dx^2}\right) dx.$ Here we have a boundary conditions given by $f(a) = f_a,\ f(b) = f_b, f'(a) = f'_a,\ f'(b) = f'_b.$

4. Let $\hat{A}$ be a square matrix, and $\vec{a}$ and $\vec{b}$ are vectors. $\vec{a}$ and λ are eigen vector and eigen value of matrix $\hat{A}$, when the following equation is satisfied.

$$\hat{A}\vec{a} = \lambda\vec{a}$$

Eigen value λ can be obtained by solving, $\det[\lambda\hat{E} - \hat{A}] = 0$, where $\hat{E}$ is the unit matrix.

(1) (計算題 5%) Obtain the eigen values and eigen vectors of a matrix $\hat{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(2) (申論題 10%) Assume that $\hat{A}$ is a Hermite matrix, meaning $(\hat{A}^T)^* = \hat{A}^{-1}$, where "*" denotes the complex conjugate. Now we consider a linear transformation of vectors $\vec{a}$ and $\vec{b}$, given by $\vec{a}' = \hat{A}\vec{a}$, and $\vec{b}' = \hat{A}\vec{b}$.
Prove $\vec{a}' \cdot \vec{b}' = \vec{a} \cdot \vec{b}$. Here $\vec{a} \cdot \vec{b} = \sum_j a_j^* b_j$ is the inner product of $\vec{a} = (a_1, a_2, \cdots)$ and $\vec{b} = (b_1, b_2, \cdots)$.

(3) (申論題 10%) Prove that the eigen values of a Hermite matrix are real.