國立臺灣師範大學 109 學年度碩士班招生考試試題

科目：**基礎數學**　　　　　　　　　　適用系所：**數學系**

注意：1.本試題共 3 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Part I : Calculus

1. (20 points) Evaluate the integrals:

   (a) $\int \sec^3(\sqrt{3}x)\,dx$ (b) $\int_{-\pi}^{\pi} \frac{2\theta\sin^2(\theta)}{\cos(4\theta)}\,d\theta$ (c) $\int_0^\infty e^{-4x^2}\,dx$ (d) $\int x^4 \sin(\sqrt{3}x)\,dx$

2. (5 points) Find the horizontal, vertical and slant asymptotes of the graph of $f(x)=\dfrac{x^3-1}{2x^2-2}$.

3. (6 points) $\displaystyle\lim_{x\to\infty} \frac{\int_0^x (\arctan x)^2\,dx}{\sqrt{4+x^2}}$

4. Find the limit.

   (a) (5 points) Prove that
   $$0<\sum_{k=1}^{n}\frac{1}{n}e^{\frac{k}{n}}-\sum_{k=1}^{n}\frac{1}{n+\frac{1}{k}}e^{\frac{k}{n}}<\sum_{k=1}^{n}\frac{1}{n^2}e^{\frac{k}{n}}.$$

   (b) (7 points) Prove that
   $$\lim_{n\to\infty}\left(\frac{e^{\frac{1}{n}}}{n+1}+\frac{e^{\frac{2}{n}}}{n+1/2}+\cdots+\frac{e}{n+1/n}\right).$$

5. (7 points) Find the maximum value of $f(x,y)=4xy$ subject to the constraint
   $$\frac{x^2}{9}+\frac{y^2}{16}=1.$$

Part II : Linear Algebra

1. Let V be a vector space over $\mathbb{R}$ with $\dim(V) \leq 5$. Suppose that $T : V \to V$ is an $\mathbb{R}$-linear operator satisfying $T(T(\mathbf{v})) = -\mathbf{v}$, $\forall \mathbf{v} \in V$. Let $\mathbf{w} \in V$ be a nonzero vector and let $W = \text{Span}(\mathbf{w}, T(\mathbf{w}))$. Suppose further that $\mathbf{u} \in V$ and $\mathbf{u} \notin W$.
   (a) (2 points) Show that $\{\mathbf{w}, T(\mathbf{w})\}$ is a basis for W.
   (b) (2 points) Show that $T(W) = W$.
   (c) (2 points) Show that $T(\mathbf{u}) \notin W$.
   (d) (4 points) Suppose $\mathbf{v} \in V$ and $a\mathbf{v} + bT(\mathbf{v}) \in W$, for some $a, b \in \mathbb{R}$. Prove $(a^2 + b^2)\mathbf{v} \in W$.
   (e) (4 points) Prove that $\{\mathbf{w}, T(\mathbf{w}), \mathbf{u}, T(\mathbf{u})\}$ is linearly independent.
   (f) (4 points) Prove that $\beta = \{\mathbf{w}, T(\mathbf{w}), \mathbf{u}, T(\mathbf{u})\}$ is a basis for V.
   (g) (2 points) Find the representation matrix for T with respect to the ordered basis β.

2. Consider the ordered basis α for $P_4(\mathbb{R})$ and β for $M_{2\times2}(\mathbb{R})$, where

$$\alpha = \{x^4 - x^3 + 2x^2, x^3 - 2x^2, x^2 + 3x + 1, x - 1, 5\},$$
$$\beta = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

   Let $T : P_4(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ be a linear transformation whose matrix representation in the ordered basis α and β is

$$\begin{pmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & -1 & -3 \end{pmatrix}.$$

   (a) (4 points) Determine the range of T.
   (b) (4 points) Determine the kernel of T.

3. (10 points) Let $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}A^7P$.

4. Let V be a vector space of functions over $\mathbb{R}$ with the basis $\beta = \{1, \sin(x), \cos(x), e^x\}$ and the inner product

$$\langle f(x), g(x) \rangle = f(0)g(0) + f'(0)g'(0) + f''(0)g''(0) + f'''(0)g'''(0),$$

   for $f(x), g(x) \in V$.

Part II : Linear Algebra

(a) (7 points) Find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ for V by applying the Gram-Schmidt method to β with respect to the given inner product such that $\mathbf{v}_1 = 1, \mathbf{v}_2 \in \text{Span}(1, \sin(x))$ and $\mathbf{v}_3 \in \text{Span}(1, \sin(x), \cos(x))$.

(b) (5 points) Find a function $f(x) \in V$ satisfying $f(0) = 1, f'(0) = 3, f''(0) = 4$ and $f'''(0) = 2$.