國立臺灣師範大學 109 學年度碩士班招生考試試題

科目：數值分析　　　　　　　　　　　　**適用系所：數學系**

注意：1.本試題共 1 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

1. Let $A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 17/4 & 11/4 \\ 1 & 11/4 & 7/2 \end{bmatrix}$.
   (a) (5 points) Is A strictly diagonally dominant? Give your reasons.
   (b) (5 points) Evaluate the matrix ∞-norm for A.
   (c) (5 points) Show that A is symmetric and positive definite.
   (d) (10 points) Find the Cholesky factorization $A = LL^\top$.
   (e) (5 points) Use the part (d) to solve the linear system $Lx = [1, 1, 1]^\top$.

2. Let f be a real-valued function defined on $[a, b]$ with $f(a)f(b) < 0$ and $f(p) = 0$ for some $p \in (a, b)$. If $\{x_n\}_{n=1}^{\infty}$ is a sequence generated by the Bisection Method, show that
   (a) (10 points) the error bounds of this method satisfy
   $$|x_n - p| \le \frac{b-a}{2^n}$$
   for each $n \ge 1$.
   (b) (10 points) the sequence of error bounds deduced in the part (a) converges linearly to 0.

3. Let $\rho(T)$ denote the spectral radius of a matrix $T \in \mathbb{R}^{n\times n}$.
   (a) (10 points) Find $\rho(T)$ for the 3×3 matrix $T = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{bmatrix}$.
   (b) (10 points) Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of vectors generated by the iteration
   $$p_{n+1} = Tp_n + c, \quad n = 0, 1, 2, \ldots,$$
   with $p_0 \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$. If $\rho(T) < 1$, show that $p_n \to p_*$ as $n \to \infty$, where p_* is the unique solution of the equation $x = Tx + c$.

4. Let $g(x) = \dfrac{2x}{3} + \dfrac{1}{x}$ for $x \neq 0$.
   (a) (10 points) Show that $g(x) \in I$ for all $x \in I = [\sqrt{3/2}, 2]$. Thus, g has at least one fixed point x_* in the closed interval I.
   (b) (5 points) Find the fixed point x_* in the part (a).
   (c) (5 points) If the sequence $\{x_n\}_{n=0}^{\infty}$ is generated by the fixed-point iteration
   $$x_{n+1} = g(x_n) = \frac{2x_n}{3} + \frac{1}{x_n}, \quad n = 0, 1, 2, \ldots,$$
   does this sequence converge to x_* for any $x_0 \in I$? Give your reasons.

5. (10 points) Find the rate of convergence of the function
   $$F(h) = \frac{\sin h}{h} + \frac{h^2}{6}$$
   as $h \to 0$.