國立高雄大學 109 學年度研究所碩士班招生考試試題

系所：應用數學系

科目：線性代數
考試時間：100 分鐘

身份別：一般生應用數學組、在職生應用數學組

是否使用計算機：否

本科原始成績：100 分

Notation.

I_n : the identity matrix of size n.
$M_{n\times m}(\mathbb{R})$: the set of $n \times m$ real matrices.

1 (32%) Let $A, B \in M_{n\times n}(\mathbb{R})$. Determine "true" or "false" for the following statements. If "true", prove it; if "false", give a counterexample.

a If rank(AB)=rank(A), then B is invertible.

b If A is similar to B, then A and B have the same characteristic polynomial.

c If $A^3 + 3A^2 + 3A + I = 0$, then A is invertible.

d If A is symmetric, then all eigenvalues of A are real.

2 Let W_1 and W_2 be the subspaces of a vector space V.
a. (10%) Show that $W_1 \cap W_2$ is a subspace of V.
b. (5%) Find $W_1 \cap W_2$, where $W_1 = \text{span}\{(1,2,3)^\top,\ (-1,2,1)^\top\}$ and $W_2 = \text{span}\{(1,-1,0)^\top,\ (2,-1,1)^\top\}$.

3 (10%) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation and

$$T(e_1) = [0,1,2]^\top,\ T(e_2) = [1,0,2]^\top,\ T(e_3) = [2,1,-3]^\top,$$

where $\{e_1, e_2, e_3\}$ is the standard ordered basis for $\mathbb{R}^3$. Let

$$\beta = \{[1,1,1]^\top, [0,1,1]^\top, [0,0,1]^\top\}$$

be an ordered basis for $\mathbb{R}^3$. Find $[T]_\beta$.

4 Let

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix}.$$

a. (10%) Find an invertible matrix Q such that $Q^{-1}AQ$ is a diagonal matrix.
b. (5%) Describe the set $W = \{p(x) | p(x) \text{ is a polynomial and } p(A) = 0\}$.

5 (10%) Let $A \in M_{n\times m}(\mathbb{R})$ with rank$(A) = m$. Show that $A^\top A$ is invertible.

6 Suppose that the characteristic polynomial of $A \in M_{6\times 6}(\mathbb{R})$ is $(3-t)^4(2-t)^2$.
a. (10%) Find all possible Jordan canonical forms of A.
b. (8%) If rank$(A - 3I) = 4$ and rank$(A - 2I) = 4$, what are the possible Jordan canonical forms of A?