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(各計算題應詳列計算推導過程，無計算推導過程者不予計分)

Notation: In the following questions, boldface letters such as **a**, **b**, etc. denote columns vectors of proper length; boldface letters such as **A**, **B** etc. denote matrices of proper size; $\mathbf{A}^T$ means the transpose of matrix **A**, and $\mathbf{A}^H$ is the Hermitian transpose (a.k.a. conjugate transpose) of **A**; $\mathbf{A}^{-1}$ means the inverse of matrix **A.** $\mathbf{I}_n$ is the $(n \times n)$ identity matrix. $||\mathbf{a}||$ means the Euclidean norm of vector **a**. By $\mathbf{A} \in \boldsymbol{R}^{m\times n}$ we mean **A** is an $m \times n$ real-value matrix. Null(**A**) is the null space, Col(**A**) the column space, and Row(**A**) the row space of **A** respectively.

一、　(10%，計算題)

Given an invertible matrix $\mathbf{A} \in \boldsymbol{R}^{n\times n}$, a nonzero matrix $\mathbf{B} \in \boldsymbol{R}^{n\times n}$, and an augmented matrix **C**=

. For each statement that follows, please answer true or false. And you **MUST** explain or prove your answer.

(A). (2%) The rank of **A** is n, and the rank of **B** is more than one.

(B). (2%) If Col(**A**)=Col(**B**), the reduced row echelon form of **C** is $[\mathbf{I}_n | \mathbf{I}_n]$.

(C). (2%) If **C** is the standard matrix of a linear transform T, T is one-to-one.

(D). (2%) $\mathbf{CC}^{\mathrm{T}}$ is invertible.

(E). (2%) Any vector **v** in $\boldsymbol{R}^n$ can be expressed as the sum of the orthogonal projection of **v** onto each column of $\mathbf{A}^{-1}$.

二、　(15%，計算題)

Let $\mathbf{A}, \mathbf{B} \in \boldsymbol{R}^{4\times 6}$, and **B** is obtained by conducting elementary row operations from **A**. **B** and the first, third, and fourth column of **A** are given by

$$\mathbf{B} = \begin{bmatrix} 1 & 3 & 0 & 1 & 4 & 2 \\ 2 & 3 & 1 & 0 & 5 & 3 \\ 2 & 3 & 0 & 1 & 5 & 2 \\ 0 & 3 & 1 & 0 & 3 & 3 \end{bmatrix}, \mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \mathbf{a}_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

(A). (4%) Determine the matrix **A** and the dimension of Null(**A**).

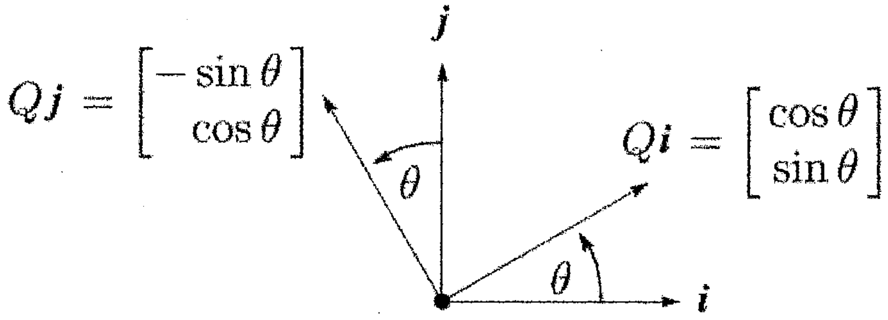
(B). (3%) Please find a basis for Row(**A**).

(C). (3%) Please find a basis for Null(**A**).

(D). (5%) Let **v**=$[1\ -1\ 0\ 0]^T$, please find the orthogonal projection of **v** on Col(**A**).

三、 (10%，計算題)

Consider the rotation by θ of real-valued vectors on a plane shown in the figure below. (Note that $0 < \theta < 2\pi$, and $\theta \neq 0$ to avoid trivial situations.)



(A). (3%) Find the transformation matrix representing the rotation.

(B). (3%) When will the transformation matrix have non-zero eigenvalues? Please explain.

(C). (4%) Is the transformation matrix similar to the identity matrix? Why?

四、 (15%，計算題)

Let $P_3 = \{c_0 + c_1x + c_2x + c_3x^3\}$ be the set of all n-th order polynomials with real-valued c_i. Define the inner product of two vectors in P_3, say $\mathbf{u} = c_0 + \ldots + c_3x^3$ and $\mathbf{v} = d_0 + \ldots + d_3x^3$, as $<\mathbf{u}, \mathbf{v}> = c_0d_0 + c_1d_1 + c_2d_2 + c_3\, d_3$. Note that P_3 can be regarded as a normed vector space.

(A). (3%) Give an example of two vectors in P_3 that are orthogonal to each other.

(B). (3%) Give an example of a unit-length vector in P_3.

(C). (3%) Show that the set $\{1 + x + x^2 + x^3,\ 1 - x - x^2 + x^3,\ 1 + x - x^2 - x^3,\ 1 - x + x^2 - x^3\}$ can be used as a basis for P_3.

(D). (3%) Use the set $\{1, x, x^2, x^3\}$ as the basis for P_3. Find the matrix representation of the linear transformation that does differentiation on polynomials in P_3.

(E). (3%) Continue with part (D). Can the matrix representing differentiation be diagonalized? Why?

五、 (10%，計算題)

Find the general solution y(x) for the given differential equations.

(A). (5%) $(x+3)^2 y'' - 8(x+3)y' + 14y = 0$

(B). (5%) $x^2 y'' - 4xy' + 6y = \ln x^2$

六、 (10%，計算題)

Find the solution y(x) of the following initial value problem for x > 1.

$$x^2 y'' - 2xy' + 2y = x \ln x, \qquad y(1) = 1, \qquad y'(1) = 0$$

七、 (15%，計算題)

Please answer the following questions:

(A). (5%) Consider a system of linear differential equations in the following vector form:

$$\frac{d}{dt}\mathbf{v}(\mathrm{t}) = \mathbf{M}\mathbf{v}(\mathrm{t})$$

Here $\mathbf{v}(\mathrm{t})$ is an n-dimensional column vector and $\mathbf{M}$ is an $n \times n$ matrix. Assume that the matrix $\mathbf{M}$ can be diagonalized by a similarity transformation:

$$\mathbf{SMS}^{-1} = \mathbf{D}$$

Here $\mathbf{S}$ is the transformation matrix and $\mathbf{D}$ is a diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its diagonal elements. The initial values of the equations form the vector $\mathbf{v}(0)$ and $\mathbf{P}(\mathrm{t})$ is a diagonal matrix with $\exp(\lambda_1 t), \exp(\lambda_2 t), \ldots, \exp(\lambda_n t)$ as its diagonal elements. Please express the solution $\mathbf{v}(\mathrm{t})$ for t > 0 in terms of $\mathbf{v}(0)$, $\mathbf{S}$ and $\mathbf{P}(\mathrm{t})$ and explain why.

(B). (5%) Continued from (A), if the matrix M is given by

$$\mathbf{M} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix},$$

write down a transformation matrix **S** that can diagonalize **M** into **D** and also write down the corresponding diagonal matrix **D.**

(C). (5%) Consider a system of differential equations in the following vector form:

$$\frac{d}{dt}\mathbf{v}(t) = \mathbf{M}\mathbf{v}(t) + \mathbf{f}(t)$$

Here the additional term $\mathbf{f}(t)$ is a known n-dimensional t-dependent column vector and the other symbols are the same as in (A). Please express the solution $\mathbf{v}(t)$ for $t > 0$ in terms of $\mathbf{v}(0)$, $\mathbf{S}$, $\mathbf{P}(t)$, $\mathbf{f}(t)$ and explain why.

八、　(5%，計算題)

Using the Laplace transform method to find the solution y(x) for the following initial value problem:

$$y''''(x) + 4y'''(x) + 11y''(x) + 14y'(x) + 10y(x) = 0,$$

$$y'''(0) = 1,\ y''(0) = 0,\ y'(0) = 0, y(0) = 0$$

[Hint: $s^4 + 4s^3 + 11s^2 + 14s + 10 = ((s+1)^2 + 1)((s+1)^2 + 4)$]

九、　(10%，計算題)

Consider the following boundary value problem in the range of $0 \le x \le 1$:

$$y''(x) + y(x) = \Lambda(x),$$
$$y(0) = 0, y(1) = 0$$

Here

$$\Lambda(x) = 1 - 2 * |x - \frac{1}{2}| \quad for \quad 0 \le x \le 1$$

(A)　(5%) $\Lambda(x)$ can be expanded into a sine series as follows:

$$\Lambda(x) = \sum_{m=1}^{\infty} c_m \sin[m\pi x]$$

Please determine the solution y(x) in the range of $0 \le x \le 1$ in terms of c_m , m=1,2,…∞.

(B)　(5%) Please determine the value of c_1.