一題十分，若無特別說明，配分由各小題平均分配。答案必須有計算過程，否則不予計分。

1. (Ordinary Differential Equation, ODE) Solve the following ODE

   (a) $xy'' + 2y' + xy = 0$, given that $\cos x/x$ is a solution

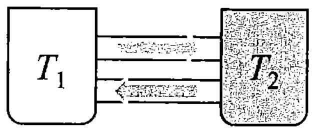
   (b) $y'' + 0.4y' + 9.04y = 0,\quad y(0) = 0,\quad y'(0) = 3$

   (c) $y'' + 2y' + y = e^{-x}$

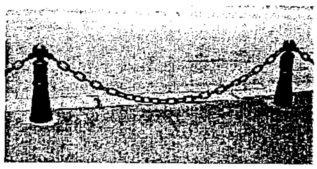
   (d) When a raindrop falls, it increases in size and the growth rate of its mass equals $km(t)$ where k is a positive constant. Applying Newton's law of motion, we get $mg = d(mv)/dt$ where g is gravitational acceleration. Find the terminal velocity of the raindrop.

2. (Applications of ODE)

   (a) Tank T_1 and T_2 contain initially 100 gal of water each. In T_1 the water is pure, whereas 150 lb of fertilizer (肥料) are dissolved in T_2. By circulating liquid between these two tanks at a rate of 2 gal/min, how long does it take before T_1 contains 50 lb of fertilizer? Assume the solution is mixed thoroughly (混合均勻) once it enters the other tank.



   (b) Show that the curve of a flexible cable hanging between two fixed points, as shown in the right figure, obeys $y'' = k\sqrt{1 + y'^2}$ where k depends on the weight. Solve for $y(x)$.



3. (Fourier series)

   (a) Find the Fourier series of a 2π-periodic function that consists of repetitions of $f(x) = |x|$ where $-\pi < x < \pi$.

   (b) Prove that $\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}$ by evaluating $\int_{-\pi}^{\pi} [f(x)]^2 dx$

4. (Forced harmonic oscillation and resonance) The motion for a harmonic oscillation under external drive obeys $m\ddot{x} = -kx - \alpha\dot{x} + F_0 \sin\omega t$. Given the initial displacement x_0 and velocity v_0, solve for $x(t)$ in the under-damped case. Find the resonance frequency ω.

5. (Vector differential calculus) Prove the following formulas for grad, div, and curl

   (a) $\nabla \times \nabla f = 0$ and $\nabla \cdot \nabla \times \vec{v} = 0$ for any scalar/vector functions $f/\vec{v}$. (3 points)

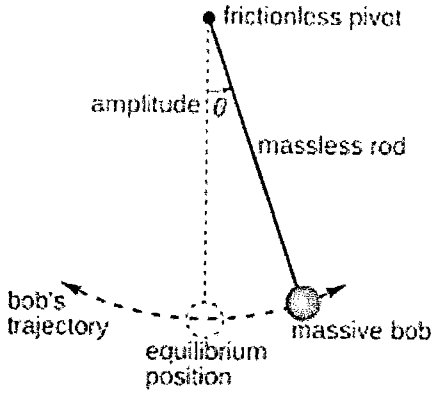
   (b) $\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \nabla^2 \vec{v}$. (4 points)

   (c) Explain what the divergence theorem and the Stokes' theorem are. (3 points)

注意:背面有試題

6. (PDE) Solve the heat diffusion equation $\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$ for a metal bar of length L with boundary $T(x=0,t)=0, T(x=L,t)=T_0$ and initial condition $T(x,t=0)=x^2$. *Useful Tip*: Set $T(x,t)=T_0\frac{x}{L}+f(x,t)$ so that the boundary condition becomes analogous to that for a vibrating string, $f(x=0,t)=0=f(x=L,t)$.

7. (Nonlinear non-homogeneous ODE) A small-amplitude pendulum of length ℓ obeys $\ell\ddot{\theta} = -g\sin\theta \approx -g\left(\theta - \frac{\theta^3}{6}\right)$. Solve this ODE by approximating $\theta^3/6$ by $\theta_0^3/6$ where θ_0 obeys $\ell\ddot{\theta} = -g\theta$ and assuming $\theta(0)=\theta_i$, $\dot{\theta}(0)=0$. *Useful tip*: $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$.



8. (Fourier transformation and Dirac delta-function) Use Fourier transform to solve Poisson's equation for a point charge $\nabla^2\phi = -4\pi\delta(\vec{r})$ in 3-dimensions. *Useful tips*: $\iiint_{-\infty}^{\infty} f(\vec{r})\delta(\vec{r})d^3\vec{r} = f(0)$, $\iiint_{-\infty}^{\infty} d^3\vec{r} = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \int_0^{\infty} r^2 dr$, and $\int_0^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}$.

9. (Laplace transform) Laplace transform is defined as $\mathcal{L}[f] \equiv \bar{f}(s) \equiv \int_0^{\infty} f(t)e^{-st}dt$.

   (a) Show that $\mathcal{L}[f] = \frac{\int_0^p f(t)e^{-st}dt}{1-e^{-ps}}$ for a function $f(t)$ with period p.

   (b) Use Laplace transform to solve $y'' + y' - 2y = 3\sin t - \cos t$; $y(0)=1, y'(0)=-1$

10. (Cauchy's integral theorem)

    (a) Integrate $\oint_C \frac{\tan\pi z}{z^2}dz$ along $16x^2+y^2=1$ clockwise.

    (b) Show that $\int_{-\infty}^{\infty}\frac{e^{px}}{1+e^x}dx = \frac{\pi}{\sin\pi p}$ for $0<p<1$. *Useful tip*: Use Cauchy formula on a rectangular contour that includes this integral and passes through $2i\pi$.