國立成功大學109學年度碩士班招生考試試題

系　所：數學系應用數學

考試科目：線性代數

考試日期：0211，節次：1

※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

A remark on notation. Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Elements of $\mathbb{F}^n$ will be regarded as column vectors when performing matrix multiplication.

1. Let $\mathcal{P}_2$ be the set of real polynomials of degree less than or equal to 2. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be defined by

$$T(a + bx + cx^2) = (a - b - c) + (3a + 2b + c)x + (b + c)x^2.$$

Discover whether T is invertible, and find the inverse in case it is. (15 points)

2. Let W be the subspace of $\mathbb{R}^5$ consisting of all $(x_1, x_2, x_3, x_4, x_5)$ which satisfy

$$\begin{aligned} 2x_1 + 2x_2 + x_3 + 2x_4 - 6x_5 &= 0 \\ 2x_1 + x_3 - 2x_5 &= 0 \\ 2x_1 - x_2 + x_3 - x_4 &= 0. \end{aligned}$$

Find a basis for W. (15 points)

3. Find A^n $(n \in \mathbb{N})$, where $A = \begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$. (15 points)

4. Consider the inner product $(x, y) \mapsto x^T A y$ on $\mathbb{R}^3$, where $A = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Let $\alpha : \mathbb{R}^3 \to \mathbb{R}$ be the linear map $\alpha(x_1, x_2, x_3) = x_1 + x_2$, and $f_1 = (1, -1, 0)$. Find $f_2, f_3 \in \mathbb{R}^3$ so that $f_2 \in \ker(\alpha)$ and (f_1, f_2, f_3) is an orthogonal (not necessarily orthonormal) basis of $\mathbb{R}^3$. (15 points)

5. Let $L : V \to W$ be a linear map between finite dimensional inner product spaces V and W.
   (a) Show that $(\operatorname{im} L)^\perp = \ker L^*$, where L^* is the adjoint of L. (10 points)
   (b) Show that $\dim \operatorname{im} L = \dim \operatorname{im} L^*$. (10 points)

6. Let $A, B \in \mathbb{C}^{n\times n}$, and $\Psi : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ be defined by $\Psi X = XA - BX$. Show that the following two conditions are equivalent. (20 points)
   (a) $\Psi X = Y$ has a solution X for every $Y \in \mathbb{C}^{n\times n}$.
   (b) A and B have no common eigenvalues.