※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. True or False $(2\% \times 15 = 30\%)$

(For the following statements, please answer **T** if it is true and **F** otherwise.)

1. Let $X_1, X_2, \ldots\ldots, X_n$ be an i.i.d. sample drawn from a population with density function $f(x;\boldsymbol{\theta})$. Furthermore, let $T_n = \sum_{i=1}^{n} X_i$, $\bar{X}_n = T_n/n$, be the summation and mean of X_i's, respectively, and $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$ be the sample variance, please answer the following questions:

   i. If the population is a normal one with unknown $\boldsymbol{\theta}=(\mu,\sigma^2)$, then $\bar{X}_n$ is a sufficient statistic for μ.

   ii. If the population is a binomial one, then $\bar{X}_n$ is a binomial random variable.

   iii. If $M(t)$ is the moment generating function of X_i, then the MGF of $\bar{X}_n$ is $M(\frac{t}{n})^n$.

   iv. $(X_1, X_2, \ldots\ldots, X_n)$ is a sufficient statistic for $\boldsymbol{\theta}$.

   v. If this population follows a Gamma distribution, then so does $\bar{X}_n$.

   vi. $E(X_i^2 X_j^2) = E(X_i^2)E(X_j^2)$ for all i,j.

   vii. If this population is a normal one then $\bar{X}_n$ and S_n^2 are independent, otherwise they are asymptotically independent.

   viii. $\bar{X}_n$ is always an unbiased estimator of the population mean regardless of the population distribution.

   ix. If the population follows a standard normal distribution, $E(X_i/X_j)=0$ for all i,j.

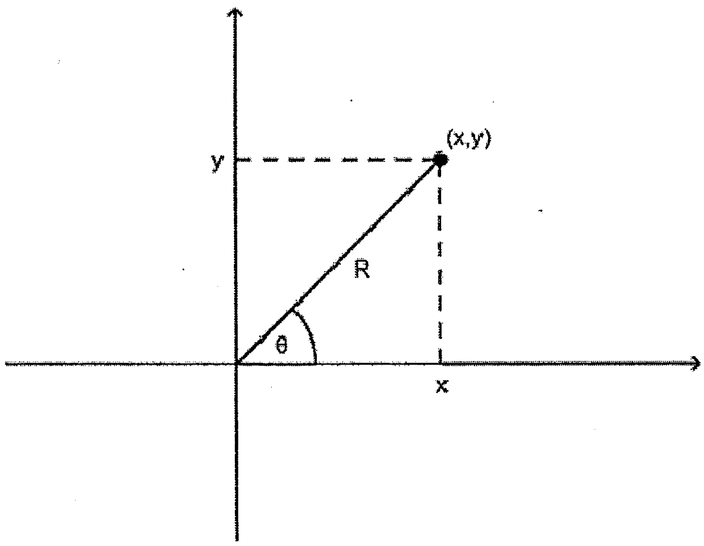
   x. If the population follows a standard normal distribution, $E(X_i^2/X_j^2)=0$ for all i,j.

2. If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, then $(X_n, Y_n) \xrightarrow{d} (X, Y)$.
3. UMVUE does not necessarily exist, but there has to be an efficient estimator.
4. Both of binomial and negative binomial distributions can converge to a Poisson distribution.
5. If X is a non-negative random variable, then $E(X)=0$ only if X is degenerate at 0.
6. The conditional variance is always less or equal to the unconditional variance.

2. Problems (70%)

1. (15%) Please prove $P(\emptyset)=0$ based on the Kolmogorov's Axioms of probability. For any result other than the probability axioms you used in the proof, you will have to prove it as well.

2. (15%) Let (X,Y) be a random vector and (R, Θ) be the corresponding coordinate of (X,Y) on the polar coordinates as shown in the figure below, where $R \geq 0, -\pi \leq \Theta \leq \pi$. Suppose that (X,Y) is uniformly distributed on a unit disc with radius one centered at the original point (0,0), that is, the range of (X,Y) is $S_{X,Y}=\{(x,y)\,|\,x^2+y^2 \leq 1\}$. Please determine the distribution of (R, Θ) and find the correlation coefficient between R and Θ.



3. (20%) Suppose that we observe a random sample $X_1, X_2, \ldots\ldots, X_n$ from a uniform distribution on the interval (0, θ), please find
   i. (10%) The MLE of θ.
   ii. (10%) The best unbiased estimator (UMVUE) of θ.

4. (20 %) Suppose that X is a discrete random variable taking values 1,2,3, and 4. The density function of X is f(x;θ), and θ takes values -1,0, and 1. The density of X is given in the table below:

x	1	2	3	4
f(x ;-1)	0.53	0.30	0.00	0.17
f(x ; 0)	0.60	0.20	0.10	0.10
f(x ; 1)	0.60	0.22	0.18	0.00

Please answer the following questions:

i. (10 %) Find the size-0.2 Likelihood Ratio Test (LRT) for testing

$$H_0: \theta = 0$$

versus

$$H_a: \theta \neq 0$$

ii. (5 %) Please comment on the test you derive in i. in terms of the power function.

iii. (5 %) Can you find a test which is more powerful/reasonable than the test you derive in i.?