

※ 注意：請於試卷內之「非選擇題作答區」標明題號依序作答。

1. (5%) Consider

$$x_1 + x_2 + \cdots + x_n = r,$$

where $a \leq x_i \leq b$ for $1 \leq i \leq n$. What is the generating function for the number of integer solutions to the above equation (where the desired count appears as the coefficient of x^r , where $r = 0, 1, \dots$)?

2. (5%) What is the number of functions from $\{1, 2, \dots, n\}^m$ to $\{1, 2, \dots, i\}^j$?

3. (10%) Consider

$$x_1 + x_2 + \cdots + x_n < r,$$

where $x_i \geq 0$ for $1 \leq i \leq n$. What is its number of nonnegative integer solutions when $n=4$ and $r=8$?

4. (10%) Derive the solution for a_n that satisfies the recurrence equation $a_n = 3a_{n-1} + n$ with $a_0 = 1$.

5. (10%) The generating function in partial fraction decomposition for the above recurrence equation is _____. (Note that expressions like

$$\frac{x-8}{(x-3)^2} - \frac{9}{x-1}$$

are *not* partial fraction decompositions.)

6. (10%) Prove the following inequality:

$$\binom{n}{\lfloor n/2 \rfloor} \geq \frac{2^n}{n}$$

where $2 \leq n$.

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7. (10%) If the polynomial function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ satisfies

$$f(-2) = 150$$

$$f(-1) = 16$$

$$f(0) = 2$$

$$f(1) = 18$$

$$f(2) = 166,$$

then a, b, c, d, e are _____, _____, _____, _____, _____, respectively.

8. (10%) The nullities of the matrices $BB^T - \lambda I$ for $\lambda = 0, 1, 2, 3, 4$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

are _____, _____, _____, _____, _____, respectively.

9. (10%) Let

$$A = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}.$$

Let

$$U = \left\{ a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

The numbers of elements $-2, -1, 0, 1, 2$ in a matrix B with

$$Bx = \begin{cases} Ax & \text{if } x \in U \\ 0 & \text{if } x \in U^\perp. \end{cases}$$

are _____, _____, _____, _____, _____, respectively.

10. (10%) If

$$A = \begin{bmatrix} 0 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

then the numbers of elements $-2, -1, 0, 1, 2$ in a matrix B with

$$ABA = A$$

$$BAB = B$$

$$(AB)^T = AB$$

$$(BA)^T = BA.$$

are _____, _____, _____, _____, _____, respectively.

11. (10%) The numbers of elements $0, 1, 2, 3, 4$ in a Jordan normal form of the matrix

$$A = \begin{bmatrix} 4 & 4 & 2 & 1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

are _____, _____, _____, _____, _____, respectively.