考試科目	基礎數學	系所別	統計學系	考試時間	2 月 7 日(五) 第一節

1. (30%) Suppose the correlation matrix of a random vector (X_1, X_2, X_3) is given as

$$\Sigma = \begin{pmatrix} 1.0 & 0.5 & -0.5 \\ 0.5 & 1.0 & 0 \\ -0.5 & 0 & 1.0 \end{pmatrix}.$$

   (a) (6%) Find the eigenvalues of Σ.

   (b) (9%) Find bases for the eigenspaces associated with each eigenvalues of Σ.

   (c) (6%) Find the diagonal matrix D and the orthogonal matrix P to orthogonally diagonalize Σ.

   (d) (3%) Find the spectral decomposition of Σ.

   (e) (6%) Show that Σ is positive semi-definite and find $\Sigma^{1/2}$.

2. (20%) Suppose A is a $n \times n$ symmetric matrix.

   (a) (10%) Show that vectors corresponding to distinct eigenspaces of A are orthogonal.

   (b) (10%) Consider the following definition. Show that

$$< \mathbf{u}, \mathbf{v} >= \mathbf{u}^T A \mathbf{v}$$

   defines an inner product in R^n.

   Definition 1 *An* **inner product** *on a vector space* $\mathbf{V}$ *is an operation that assigns to every pair of vectors* $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ *a real number* $< \mathbf{u}, \mathbf{v} >$ *such that the following properties hold for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$ *and* $c \in R$.

   i. $< \mathbf{u}, \mathbf{v} >=< \mathbf{v}, \mathbf{u} >$.

   ii. $< \mathbf{u}, \mathbf{v} + \mathbf{w} >=< \mathbf{u}, \mathbf{v} > + < \mathbf{u}, \mathbf{w} >$.

   iii. $< c\mathbf{u}, \mathbf{v} >= c < \mathbf{u}, \mathbf{v} >$.

   iv. $< \mathbf{u}, \mathbf{u} >\geq 0$ *and* $< \mathbf{u}, \mathbf{u} >= 0$ *if and only if* $\mathbf{u} = \mathbf{0}$.

   A vector space with an inner product is called an **an inner product space**.

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。

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3. (5 %; 1% for each part) Write down the final answers only.

   (a) Find $\frac{d}{dx}(x^2+3x+1)$.

   (b) Find $\frac{d}{dx}e^x$.

   (c) Find $\frac{d}{dx}(\ln(x)+x\sin(x))$.

   (d) Find $\frac{d}{dx}\frac{1}{2+\cos(x)}$.

   (e) Find $\frac{d}{dx}\sin(\cos(x))$.

- Note. For Problems 4 – 7, you need to show your work in the solutions. Writing down the final answers only for these problems is not enough to receive any points.

4. (20%) Let $I_n(t)=\int_0^t x^n\cos(x)dx$ and $J_n(t)=\int_0^t x^n\sin(x)dx$ for $n\geq 0$ and $t\in(-\infty,\infty)$.

   (a) (14%) Express $I_{n+1}(t)$ and $J_{n+1}(t)$ in terms of $I_n(t)$, $J_n(t)$, n and t.

   (b) (6 %) Compute $I_0(\pi)$ and $J_0(\pi)$, and then find $I_1(\pi)$ and $J_1(\pi)$ using the expressions in Part (a).

5. (10 %) Let $D(r)=\{(x,y):x^2+y^2\leq r^2\}$ for $r>0$.

   (a) (7%) Find $\int_{D(r)} e^{-x^2-y^2}d(x,y)$ for $r>0$.

   (b) (3 %) Find $\lim_{r\to\infty}\int_{D(r)} e^{-x^2-y^2}d(x,y)$ based on your answer for Part (a).

6. (10 %) Let $f(x,y)=x^2+\sin(xy)+y^2$. Does f have a local minimum or a local maximum at $(x,y)=(0,0)$? Justify your answer.

7. (5%) Let $\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ for $x\in(-\infty,\infty)$ and $G(x)=\int_x^\infty \phi(t)dt$ for $x\in(-\infty,\infty)$. Find $\lim_{x\to\infty}\frac{xG(x)}{\phi(x)}$.

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。