第一單元: 離散數學 (本單元共計 60 分)

1. (10 %) Let A_n ($n \geq 2$) be the number of strictly increasing sequence of positive integers that have 1 as the first term and n as the last term. That is, sequences $x_1, x_2, \ldots, x_k$ where $x_1 = 1$ and $x_k = n$ and $x_j < x_{j+1}$ for $1 \leq j < k$.
   (a) What are A_2 , A_3 and A_4 ?   [3%]
   (b) Find a recurrence relation for A_n ($n > 1$).   [5 %]
   (c) Solve the correct recurrence relation you obtained at (b) with A_2 as the initial condition for A_n ($n \geq 2$).   [2%]

2. (15%) The rooted Fibonacci trees T_n are defined recursively as follows:
   (Basis:) T_1 and T_2 are both rooted trees consisting of a single vertex.
   (Recursion:) T_n ($n > 2$) is constructed from a root with T_{n-1} as its left subtree and T_{n-2} as its right subtree.
   (a) What is the number of leaves of T_n for $n > 1$?   [5%]
   (b) What is the number of internal vertices of T_n for $n > 1$?   [5%]
   (c) What is the height of T_n for $n > 1$?   [5%]

3. (10 %) Let A be a set with 10 elements. Then
   (a) How many ternary relations are there on A ?   [3%]
   (b) How many asymmetric relation are there on A ?   [3%]
   (c) What is the number of all simple graphs which have A as the vertex set ?
       [4 %]

4. (10 %) Show that in a connected simple graph with $n > 0$ vertices, there must exist a path of length $< n$ between every two distinct vertices.

5. (15 %) Two sets A and B are said to be equipotent (, written $A \sim B$,) if there is a 1-1 and onto mapping between them. Let N be the set of positive integers and K = $\{x \mid x$ is a real number and $0 \leq x \leq 1\}$. Show that
   (a) $\sim$ is an equivalence relation on the class of all sets, [5%]
   (b) $K \sim K^2$ where $K^2 =_{def} \{(x,y) \mid x, y \in K\}$, [5%]
   (c) K and N are not equipotent.   [5%]
   Hint for (b) and (c) : Each number in K can be represented *uniquely* as an infinite sequence of digits : $0. x_1 x_2 x_3 \ldots$. For example, $1/3 = 0.3333333\ldots$ and $1/2 = 0.500000\ldots$. This means K and the set $\{f \mid f$ is a function from N to $\{0,\ldots,9\}\}$ are equipotent.

科目 計算機數學（機率）
Course
班級
學號 姓名
日 Date,
節 Period
教務編號
CourseNo.

一、設隨機變數 X 和 Y 的結合密度函數為
$f(x,y) = kx^2y$，其中 k 是待定常數，$0 \le x, y \le 1$
(6分)(1)求 k 值.
(6分)(2)求 $P(Y \ge X | Y \ge \frac{1}{2})$.

二、設隨機變數 X 和 Y 的結合密度函數為
$f(x,y) = 3x$ ， $0 < y < x < 1$.
(6分)(1)求 Y 的邊際密度函數 (marginal density function).
(6分)(2)給定 $Y = \frac{1}{2}$，求 X 的條件密度函數 (conditional density function).
(6分)(3)給定 $Y = \frac{1}{2}$，求 X 的條件變異數 (conditional variance).

三、設隨機變數 X 在區間 $(0,1)$ 上具有均勻分配
(10分) (uniform distribution). 求 $Y = -\frac{1}{2}\ln(1-X)$ 的密度函數 (density function).