考試科目	線性代數	所別	應用數學	考試時間	4月21日 星期六 下午第二節

說明：

一. 總共六大題，每大題含五小題（分別依編號 [1],[2],[3],[4],[5] 標示於題句中），每小題計4分（亦即每大題為20分），共120分。

二. 作答時，大題中之小題不可顛倒順序。凡答案不完全正確，該小題即不予計分。得分如超過100分，則仍以100分計。

三. 在答案卷上，請清楚地標明題號及簡潔的答案（演算或證明步驟不必列出），如下例：

1. [1] $p=\begin{bmatrix}1\\2\end{bmatrix}$, $q=\begin{bmatrix}3\\4\end{bmatrix}$
   [2] $P=\begin{bmatrix}1&2\\3&4\end{bmatrix}$
   [3] $x=[1,2,3,4,5,6]^T$
   [4] 100
   [5] $y=20t+80$

4. [1] $\lambda_1=10,\ \lambda_2=20$
   [2] $x_1=\begin{bmatrix}1\\2\end{bmatrix}$, $x_2=\begin{bmatrix}3\\4\end{bmatrix}$
   [3] $P_1=\begin{bmatrix}1&2\\3&4\end{bmatrix}$, $P_2=\begin{bmatrix}5&6\\7&8\end{bmatrix}$
   [4] True
   [5] Yes, A is positive definite because ....

1. (a) Given a (column) vector $\mathbf{b}=[1,2,0,3]^\mathsf{T}$ and a subspace $S: x_1-x_2+x_3-x_4=0$ in $\mathbb{R}^4$, find the projection vectors $\mathbf{p}$ and $\mathbf{q}$ [1] so that $\mathbf{b}=\mathbf{p}+\mathbf{q}$, where $\mathbf{p}\in S$ and $\mathbf{q}\in S^\perp$. Also determine the projection matrix P [2] onto S so that $P\mathbf{b}=\mathbf{p}$. (b) Given data (t_j,b_j), $j=1,2,\ldots,m$, we assume $f(t_j)=x_1\phi_1(t_j)+x_2\phi_2(t_j)+\cdots+x_n\phi_n(t_j)\approx b_j$ (a linear model), $m>n$, or, using matrix/vector notation, $A\mathbf{x}\approx\mathbf{b}$, where $A=[a_{jk}]\equiv[\phi_k(t_j)]\in\mathbb{R}^{m\times n}$, $\mathbf{x}=[x_1,x_2,\ldots,x_n]^\mathsf{T}\in\mathbb{R}^n$, and $\mathbf{b}=[b_1,b_2,\ldots,b_m]^\mathsf{T}\in\mathbb{R}^m$. Suppose the measurements are given as

t_j	−2	−1	0	1	2
b_j	4	2	−1	0	0

Assume $\phi_1(t)=1$, $\phi_2(t)=t$; find the $\mathbf{x}\in\mathbb{R}^2$ [3] that minimizes $\|A\mathbf{x}-\mathbf{b}\|_2$; the minimal value of $\|A\mathbf{x}-\mathbf{b}\|_2$ is [4]. In other words, the *best line* $y=ct+d$ to fit the data is [5], with the minimal sum of the squares of errors $\sum_{j=1}^{m}(ct_j+d-b_j)^2=\|A\mathbf{x}-\mathbf{b}\|_2^2$.

2. (a) A matrix $A\in\mathbb{C}^{n\times n}$ is diagonalizable if there exists a nonsingular matrix $X\in\mathbb{C}^{n\times n}$ such that $X^{-1}AX=D$ is diagonal. The columns of X are the [1] of A, and the elements of D are the [2] of A. (b) Specify a further property on X [3] if A is *normal* (i.e., $A^*A=AA^*$, where A^* is the conjugate transpose of A). In this normal class of matrices, give the *additional* property that the elements of D have if A is : (i) Hermitian ($A^*=A$) [4]; (ii) unitary ($A^*A=I$) [5].

備考 | 試題隨卷繳交

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3. (a) Let $A^{\mathsf{T}} = A \in \mathbb{R}^{n\times n}$ with eigenvalues $\{\lambda_j\}_{j=1}^n$ ordered so that $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, and let $\mathbf{x} = [x_1, x_2, \ldots, x_n]^{\mathsf{T}} \in \mathbb{R}^n$. Express the quantities p and q [1] in terms of the eigenvalues, where $p = \min\{\mathbf{x}^{\mathsf{T}}A\mathbf{x} : ||\mathbf{x}||_2 = 1\}$ and $q \equiv \max\{\mathbf{x}^{\mathsf{T}}A\mathbf{x} : ||\mathbf{x}||_2 = 1\}$; do the same for r [2], where $r \equiv \max\{||A\mathbf{x}||_2 : ||\mathbf{x}||_2 = 1\}$. (b) Given $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$, find an orthogonal matrix Q [3] such that $Q^{\mathsf{T}}AQ$ is diagonal; determine the projection matrices P_1 and P_2 [4] such that $A = \lambda_1 P_1 + \lambda_2 P_2$, where λ_1, λ_2 are the eigenvalues of A; also compute the matrices $P_1^4 + P_2^4$ and $P_1^4 P_2^4$ [5].

4. Suppose $\rho_{k+2} = \frac{1}{2}\rho_{k+1} + \frac{1}{2}\rho_k$ for $k = 0, 1, 2, \ldots$, where $\rho_0 = 0$ and $\rho_1 = 1$. (a) Let $\mathbf{r}_k = [\rho_k, \rho_{k+1}]^{\mathsf{T}}$. Find the matrix A [1] so that $\mathbf{r}_{k+1} = A\mathbf{r}_k = A^{k+1}\mathbf{r}_0$. (b) Compute the eigenvalues [2] and eigenvectors [3] of A. (c) Determine $\lim_{k\to\infty} A^k$ [4] and $\lim_{k\to\infty} \rho_k$ [5].

5. (a) If $A^{\mathsf{T}} = A \in \mathbb{R}^{n\times n}$ is positive definite, then $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathsf{T}}A\mathbf{x} - \mathbf{x}^{\mathsf{T}}\mathbf{b}$ assumes its minimum at the point $\mathbf{x} =$ [1]. Given the quadratic $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_1 - x_3$, with $\mathbf{x} = [x_1, x_2, x_3]^{\mathsf{T}} \in \mathbb{R}^3$, find A and determine if A is positive definite [2]. Does f have a minimum? (If so, give the point $\mathbf{x}$ where it happens.) [3] (b) Determine whether the following matrix A is positive definite by factoring it as $A = R^{\mathsf{T}}R$, where R is an upper triangular matrix [4]:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 2 & 8 & 6 & 8 & 4 \\ 2 & 6 & 9 & 10 & 7 \\ 2 & 8 & 10 & 13 & 8 \\ 1 & 4 & 7 & 8 & 6 \end{bmatrix}.$$

For this matrix, does there exist an $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{x}^{\mathsf{T}}A\mathbf{x} = 0$ [5]?

6. Given a matrix $M \in \mathbb{C}^{n\times n}$, let J be called the Jordan matrix of M in the canonical form $M = SJS^{-1}$. True or false: Two matrices are similar if they have the same characteristic polynomial and the same minimal polynomial [1]? Suppose an 8-by-8 matrix A has the following properties: $\text{rank}(A) = 5$, $\text{rank}(A^2) = 2$, $\text{rank}(A^k) = 1$ for $k \ge 3$, and $\text{trace}(A) = 2$. (a) Determine the Jordan matrix of A [2]. (b) Give the minimal polynomial of A [3]. (c) Write out the companion matrix C of the characteristic polynomial of A [4]. (d) Determine the Jordan matrix of C [5].

備考	試題隨卷繳交