考試科目	微積分	所別	數學	考試時間	4月20日上午第一節 星期六

1. (*a*) Show that for every real x the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges.(5%)

   (*b*) Denoting $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$, then $f(x)$ is continuous in $[0, \pi]$. (5%)

   (*c*) Prove that $\int_0^{\pi} f(x)dx = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}$. (5%)

2. Evaluate the limits

   (*a*) $\lim_{x \to 0^+} \left(\ln \frac{1}{x}\right)^x$. (7%)

   (*b*) $\lim_{h \to 0} \frac{1}{h} \int_h^{2h} \left(\frac{\sin^{-1} x}{x}\right)^{\frac{1}{x^2}} dx$. (10%)

3. Let $F(x) = \int_0^x f(t)dt$. Determine a formula (or formulas) for computing $F(x)$ for all real x, if f is defined as follows

   (*a*) $f(t) = \frac{2t+5}{t^2+2t-3}$. (7%) (*b*) $f(t) = \frac{1}{\cos t + \sin t}$. (8%)

4. Let $f(x, y, z) = x^2 + y^2 + z^2$.

   (*a*) Find an equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 6$. (5%)

   (*b*) What is the maximum rate of increase of f at $(1, -1, 2)$. (5%)

   (*c*) Find the minimum of the values $\frac{|x - y + 2z|}{\sqrt{6}}$ on the sphere $f(x, y, z) = 6$. (8%)

5. Calculate the iterated integral $\int_0^1 \int_0^y (x \sin x + y^2 \cos x)dxdy$. (10%)

6. Prove or disprove the following statements.

   (*a*) If f_n and f are C^1 – functions and f_n converges uniformly to f in S, then f_n' converges to f' in S. (10%)

   (*b*) Let f and g be two functions continuous in a closed interval $[a, b]$ and having derivatives in open interval (a, b). Then there exists c in (a, b) such that

   $$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)). \quad (15\%)$$