考試科目	微積分	系所別	應用數學系	考試時間	2月5日(星期五)第三節

Show all your work and carefully justify all your answers. Answers without explanation will not receive any score.

1. (14 points) Consider the following function:

$$f(x) = \begin{cases} 3x - 1, & x < 1; \\ 2x, & x \geq 1. \end{cases}$$

Use the ϵ-δ definition of the limit to show that $\lim_{x \to 1} f(x) = 2$.

2. Consider

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(a) (3 points) Find the partial derivatives at $(0,0)$ and the gradient vector $\nabla f(0,0)$.

(b) (6 points) Use the definition of the directional derivative to find $D_{\mathbf{u}}(0,0)$ for all unit vectors $\mathbf{u} = \langle a, b \rangle$.

(c) (6 points) Is f continuous at $(0,0)$? How about differentiability?

3. (a) (6 points) Let $f(x)$ be a continuous function. Show that if $f(x)$ has two local maxima, it also must have a local minima.

(b) (8 points) Show that part (a) becomes wrong for a function of two variables by finding all local maxima and minima of the function:

$$f(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2.$$

4. (a) (7 points) Evaluate

$$\int \frac{x}{(x+1)(x+2)} \mathrm{d}x.$$

(b) (7 points) Evaluate

$$\int_0^1 \frac{\mathrm{d}x}{(2-x)\sqrt{1-x}}.$$

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。

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5. (a) (7 points) Use a suitable change of variable to compute

$$\iint_R (x+y)^2 e^{x^2-y^2}\,\mathrm{d}A,$$

where R is the square with vertices $(1,0), (0,1), (-1,0)$ and $(0,-1)$.

(b) (7 points) Evaluate

$$\iiint_{x^2+y^2+z^2\le 1} e^{(x^2+y^2+z^2)^{3/2}}\,\mathrm{d}V.$$

6. Consider the series:

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)(n+2)}.$$

(a) (3 points) Find radius and interval of convergence.

(b) (6 points) Show that for $|x|<1$:

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)(n+2)} = \frac{1-x}{x^2}\ln(1-x)+\frac{1}{x}. \qquad (1)$$

(c) (6 points) The function on the right-hand side of (1) has a removable discontinuity at $x=1$. Remove it and then show that (1) also holds for $x=1$.

7. (a) (6 points) Consider

$$f(x) = \int_{\tan x}^{x/4} \sin(t^2)\,\mathrm{d}t.$$

Compute $f'(\pi)$.

(b) (8 points) Suppose that $f(0)=0$ and

$$f'(\ln x) = \begin{cases} 1, & \text{if } 0 < x \le 1; \\ x, & \text{if } 1 < x < \infty. \end{cases}$$

Find $f(x)$.

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