※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

一. Linear Algebra (50%)

1. Two eigenvectors of this circulant matrix $\mathbf{C}$ are $(1,1,1,1)^T$ and $(1,i^3,i^6,i^9)^T$. What are the eigenvalues λ_0 and λ_1? (15%)

$$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \lambda_0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ i^3 \\ i^6 \\ i^9 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ i^3 \\ i^6 \\ i^9 \end{bmatrix}$$

2. Let

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

(a) Compute the LU factorization of $\mathbf{A}$. (10%)

(b) Explain why $\mathbf{A}$ must be positive definite. (10%)

3. Let $p(x) = -x^3 + cx^2 + (c+3)x + 1$, where c is a real number. Let

$$\mathbf{C} = \begin{bmatrix} c & c+3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and let

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & -c-3 \\ 1 & -1 & c+2 \\ -1 & 1 & -c-1 \end{bmatrix}$$

(a) Compute the $\mathbf{A}^{-1}\mathbf{C}\mathbf{A}$. (10%)

(b) Show that $\mathbf{C}$ is the companion matrix of $p(x)$ and use the result from part (a) to prove that $p(x)$ will have only real roots regardless of the value of c. (5%)

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二、Discrete Mathematics (50%) (請說明如何求解過程,只寫答案不予計分)

4. (15%) Consider the set S={a,b,c,d,e}, and a logical matrix M corresponding to a relation R on the set S is shown in the following.

$$M=\begin{bmatrix}1&0&1&0&0\\0&1&0&0&0\\0&0&1&0&0\\1&1&1&1&1\\0&1&1&0&1\end{bmatrix}$$

(a) (5%) In order to check if there exists a partial ordering, which properties should you check?

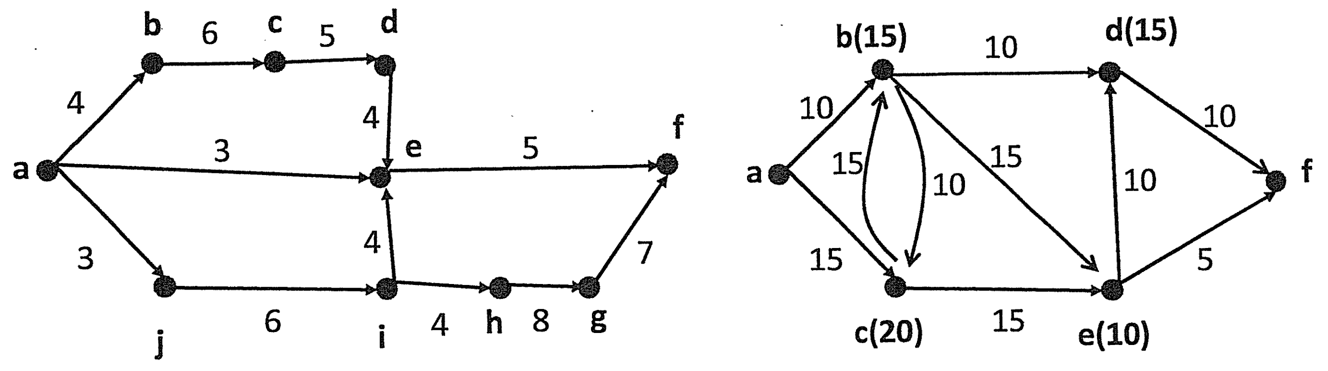
(b) (5%) Draw the Hasse diagram of this order.

(c) (5%) Determine all pairs of incomparable elements.

5. (15%)Solve the following recurrence equations for *A(n)*, and *B(n)*.

$$\begin{cases}A(n)=3A(n-1)+2B(n-1)\\ B(n)=A(n-1)+B(n-1)\\ n\geq 1, A(0)=\sqrt{3}, B(0)=0\end{cases}$$

6. (20%) Given the following figures (a) and (b). Assume the source node is a and the sink node is f. The label along the edge (or node) means the maximum capacity of the edge (or node).



(a)　　　　　　　　(b)

(a) (10%) Find the maximum flow from the source to the sink. And what is the corresponding minimum cut in Fig. (a).

(b) (10%) Find the maximum flow from the source to the sink. And what is the corresponding minimum cut in Fig. (b)