1. (25 pts) Solve the following equations.
   (a) $\dfrac{dy}{dx} = \dfrac{y^2}{xy - x^2}$.
   (b) $\dfrac{dy}{dx} - \dfrac{a(x)}{x} y = x^3, x > 0; \quad y(1) = 1,$

   where

$$a(x) = \begin{cases} 2 \text{ for } 0 < x \leq 2, \\ 3 \text{ for } x > 2. \end{cases}$$

   (Assume $y(x)$ is continuous and piecewise differentiable.)

2. (25 pts) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & -5 & -3 \\ 3 & 6 & 4 \end{pmatrix}.$$

   (a) Find $e^{\mathbf{A}t}$.
   (b) Solve $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

3. (25 pts)
   (a) Assume $\beta > 0$. Solve the equation $y'' + y = \sin(\beta t),\ y(0) = 1,\ y'(0) = 0$.
   (b) Find the limit $\lim_{\beta \to \infty} y(t)$.
   (c) Let $f(t)$ be a continuous function satisfying $f(t + 2\pi) = f(t)$ and $f(-t) = f(t)$. Assume that $z(t)$ is the solution of $z'' + z = f(\beta t),\ z(0) = 0,\ z'(0) = 2$. Find $\lim_{\beta \to \infty} z(t)$.

4. (25 pts) Let
   $V_1 = \{y(t) \,|\, y(t) \text{ is a soution of } y'' - ty' + 2y = 0, 0 \leq t \leq 5\}$,
   $V_2 = \{z(t) \,|\, z(t) \text{ is a soution of } z''' + (t^2 - 2)z'' + tz' - 4z = 0, 0 \leq t \leq 5\}$,
   $V_3 = \{y(t) + z(t) \,|\, y(t) \in V_1, z(t) \in V_2\}$.
   Find the dimensions of V_1, V_2 and V_3. Explain your reasons.

試題隨卷繳回