國立中央大學108學年度碩士班考試入學試題

所別：天文研究所 碩士班 不分組(一般生)　　　　共2頁　第1頁

科目：應用數學

本科考試禁用計算器

(1) (20 points)
   (a) (10 points) Does the equation $y = x^3 - 3x + 2 = (x+2)(x-1)^2$ has a maximum or a minimum? If yes, find out their locations and values. With these information and the location of the root(s), sketch the plot y vs x of the equation.
   (b) (6 points) Similarly, sketch the curve $y^2 = x^3 - 3x + 2$. (Note that the left hand side is y^2.)
   (c) (4 points) Without going through lengthy calculation, roughly sketch the curve $y^2 = x^3 - 3x + b$ for (i) b is a little bit smaller than 2, say 1.9; and (ii) b is a little bit larger than 2, say 2.1.

(2) (20 points)
The relation between Cartesian coordinates (x, y, z) and spherical coordinates (r, θ, ϕ) are $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. Let $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$ be the orthonormal basis (i.e., orthogonal unit vectors) of the Cartesian coordinate system, and $\{\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi\}$ be the orthonormal basis in spherical coordinate system in the direction of increasing r, θ, ϕ, respectively.
   (a) (4 points) Show the relationship of the two coordinate systems in a figure.
   (b) (6 points) Express $\{\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi\}$ in terms of $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$ and the spherical coordinates.
   (c) (10 points) Express the position vector of a point particle in spherical coordinate system. Find the velocity and acceleration components of the particle in spherical coordinates (r, θ, ϕ).

(3) (20 points)
Consider the matrix

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

   (a) (10 points) Find the eigenvalues and the corresponding eigenvectors of the matrix.
   (b) (2 points) Are there any pairs of eigenvectors perpendicular to each other? If yes, what are they?
   (c) (8 points) Find the inverse of the matrix.

(4) (20 points)
   (a) (5 points) A force can be expressed as the gradient of a potential, i.e., $\mathbf{F} = -\nabla\Psi$ is called a conservative force. Show that $\nabla \times \mathbf{F} = 0$.
   (b) (10 points) The centre of the Earth is located at the origin of a Cartesian coordinate system. The centre of the Moon is located on the z-axis at a fixed distance R from the origin. The tidal force exerted by the Moon on a point mass at the surface of the Earth (x, y, z) can be approximated by (when x, y, z are much smaller than R)

$$F_x = -GMm\frac{x}{R^3}, \quad F_y = -GMm\frac{y}{R^3}, \quad F_z = GMm\frac{2z}{R^3},$$

   where m is the mass of the Moon and M is the mass of the Earth. Show that the tidal force is a conservative force, and work out the corresponding potential.
   (c) (5 points) Sketch the contours of the potential surfaces on the $x-z$ plane. Don't forget to label the relative levels of the contours (e.g., which one corresponds to is higher potential and which one to lower).

注意：背面有試題

參考用

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(5) (20 points)
Given an inhomogeneous ODE (ordinary differential equation)

$$\ddot{x} + \frac{\dot{x}}{t} - \frac{x}{t^2} = \delta\left(t - t_0\right) ,$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}t$, $\delta(\xi)$ is the Dirac delta function and t_0 is a constant.

(a) (5 points) If you are seeking continuous solution for $x(t)$, what kind of matching condition(s) (or boundary condition(s)) will you suggest for x at time $t = t_0$. Please state your reasons.

(b) (15 points) Consider $t_0 > 0$. If the initial conditions for x is $x(0) = 0$ and $\dot{x}(0) = 1$, solve the ODE for x in the two domains (i) $0 < t < t_0$ and (ii) $t_0 < t$ with the matching condition(s) you suggested in (a).

參考用

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