國立臺北大學 108 學年度碩士班一般入學考試試題

系（所）組別：統計學系

科　　目：數理統計

☐可 ☑不可使用計算機

I. （50%）

1. (10%) Let X and Y denote independent random variables with respective probability density function $f(x) = 2x, 0 < x < 1$, zero elsewhere, and $g(y) = 3y^2, 0 < y < 1$, zero elsewhere. Let $U = \min(X, Y)$ and $V = \max(X, Y)$. Find the joint probability density function of U and V.
2. (40%) Let $X_1, X_2, \cdots, X_n$ be a random sample from the distribution with a probability density function

$$f(x;\theta) = \theta x^{\theta-1}, 0 < x < 1;\ \theta > 0.$$

Let the distribution function be denoted as $F(x;\theta)$. For $0 < p < 1$, define the pth quantile of this distribution to be $\xi_p = F^{-1}(p)$, i.e., $P[X_i \le \xi_p] = p$.

(a) (10%) Find the maximum likelihood estimator $\hat{\theta}$ of θ and the distribution of $\hat{\theta}$.

(b) (5%) Find ξ_p and the corresponding maximum likelihood estimator $\hat{\xi}_p$ of ξ_p.

(c) (10%) Find the asymptotic distribution of $\hat{\xi}_p$.

(d) (10%) Let $0 < X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ denote the order statistics $X_1, X_2, \cdots, X_n$. Find $E[F(X_{(k)})]$, where $k \doteq p(n+1)$. Based on the result, find another estimator $\tilde{\xi}_p$ of ξ_p?

(e) (5%) Suppose a random sample of size 49 is collected as follows:

0.06 0.06 0.09 0.15 0.17 0.17 0.18 0.21 0.23 0.29 0.29 0.31 0.34 0.34 0.37
0.41 0.41 0.42 0.44 0.45 0.46 0.47 0.47 0.49 0.49 0.49 0.50 0.53 0.56 0.57
0.57 0.58 0.58 0.59 0.59 0.60 0.63 0.64 0.66 0.68 0.68 0.71 0.74 0.78 0.81
0.85 0.85 0.91 0.97.

We obtain $\sum_{i=1}^{49} x_i = 23.84$ and $\sum_{i=1}^{49} \log(x_i) = -43.052$. Find $\tilde{\xi}_{0.6}$ and $\hat{\xi}_{0.6}$.

II. （50%）

Let $X_1, X_2, \ldots, X_n$ be a random sample from $\mathrm{UNIF}[0, \theta]$.

1. (10%) Derive the Maximum Likelihood Estimator (MLE) of θ.
2. (10%) Is your MLE unbiased? Prove or disprove the unbiasedness.
3. (10%) Calculate the variance of your MLE. Compare it to the Cramér-Rao Lower Bound and comment on your observation.
4. (10%) Is your MLE consistent? Prove or disprove the consistency.
5. (10%) Use your MLE to conduct a hypothesis testing of

$$H_0: \theta = 2 \text{ versus } H_1: \theta > 2 .$$

Given that the size of the test is 0.05, find the rejection region.

試題隨卷繳交