考試科目	線性代數	所別	應用數學系	考試時間	3月15日 星期六 第2節

Please show all your work.

1. Define $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$.

   (a)Let $P^{-1}AP = D$ be a diagonal matrix. Prove that $e^A = Pe^D P^{-1}$. (10%)

   (b) Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Compute e^A. (7%)

2. Label the following statements as true or false. In each part, V and W are finite-dimensional vector spaces (over F), A, B are matrices.

   (a)If $T, U : V \to W$ are both linear and agree on a basis for V, then $T=U$.

   (b)If $m = \dim(V)$ and $n = \dim(W)$, β, γ are ordered basis of V and W, respectively, and T is a linear transformation, then $[T]_\beta^\gamma$ is an $m \times n$ matrix.

   (c) $A^2 = I \Rightarrow A = I$ or $A = -I$.

   (d) $AB = I$ implies that A and B are invertible.

   (e) Let T be a linear operator on a finite-dimensional vector space V. Let β and α be ordered basis of V, and let Q be the change of coordinate matrix that changes α-coordinates into β-coordinates. Then $[T]_\beta = Q[T]_\alpha Q^{-1}$. (20%)

3. Let $A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{bmatrix}$

   (a)Find the characteristic polynomial of A. (6%)

   (b)Find a Jordan canonical form J and an invertible matrix Q such that $J = Q^{-1}AQ$. (10%)

4. A matrix $M \in M_{n\times n}(C)$ is called skew-symmetric if $M^t = -M$.

   Prove that if M is skew-symmetric and n is odd, then M is not invertible.
   What happens if n is even? (15%)

5. (a) Let $V = P(R)$ with the inner product $<f, g> = \int_{-1}^{1} f(t)g(t)dt$. Use Gram-Schmidt process to obtain an orthonormal basis for $P_2(R)$ from the standard basis $\{1, x, x^2\}$. (10%)

   (b) Let $V = P_3(R)$ with the inner product $<f, g> = \int_{-1}^{1} f(t)g(t)dt$. Compute the orthogonal projection of $f(x) = x^3$ on $P_2(R)$. (7%)

6. Let F be a field that is not of characteristic 2.

   Define $W_1 = \{A \in M_{n\times n} : A_{ij} = 0$ whenever $i \le j\}$ and W_2 to be the set of all symmetric $n \times n$ matrices with entries from F. Both W_1 and W_2 are subspaces of $M_{n\times n}(F)$. Prove that $M_{n\times n}(F) = W_1 \oplus W_2$. (15%)