淡江大學 108 學年度碩士班招生考試試題

系別：數學學系　　　　　　科目：線性代數　　　　　　23-1

考試日期：3 月10日 (星期日) 第 2 節　　　　　　本試題共 6 大題，共 1 頁

1. Let $A = \begin{bmatrix} 2 & 3 & 1 & 4 & 5 \\ 3 & 2 & 4 & 5 & 3 \\ 2 & 1 & 3 & 3 & 1 \\ 3 & 1 & 5 & 4 & 0 \end{bmatrix}$.

   (1) (8%) Find the reduced row echelon form of A, rank of A, and nullity of A.

   (2) (12%) Find bases of the row space, column space, and null space of A respectively.

2. Let $A = \begin{bmatrix} 4 & 0 & -3 \\ -1 & 5 & 9 \\ 0 & -2 & -3 \end{bmatrix}$. Define the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$.

   (1) (8%) Find all eigenvalues and eigenvectors of T.

   (2) (4%) Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

3. (12%) Define an inner product $\langle \cdot, \cdot \rangle$ on the real vector space $\mathbb{R}^3$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + 2x_3y_3, \text{ for all } \mathbf{x} = (x_1, x_2, x_3) \text{ and } \mathbf{y} = (y_1, y_2, y_3) \text{ in } \mathbb{R}^3.$$

   Use Gram-Schmidt process to construct an orthogonal basis from the basis

$$\beta = \{(1,1,2), (-2,0,3), (2,10,2)\}$$

   of $\mathbb{R}^3$.

4. (16%) Let $A = \begin{bmatrix} -2 & 0 & 4 & 6 \\ 0 & -2 & -2 & -3 \\ 0 & 0 & -14 & -16 \\ 0 & 0 & 9 & 10 \end{bmatrix}$. Find matrices P and J such that $J = P^{-1}AP$ is the Jordan canonical form of A.

5. Let A be an $m \times n$ matrix and B be an $n \times m$ matrix.

   (1) (10%) Prove that $\det(I_m + AB) = \det(I_n + BA)$.

   (2) (10%) Prove that AB and BA have the same nonzero eigenvalues (counting algebraic multiplicity).

6. Let $A = [a_{ij}]$ be an $n \times n$ matrix with $a_{ij} \in \mathbb{C}$ for all $1 \leq i \leq n$ and $1 \leq j \leq n$. Define $r_i = \sum_{\substack{1 \leq j \leq n \\ j \neq i}} |a_{ij}|$ and $D_i = \{z \in \mathbb{C} : |z - a_{ii}| \leq r_i\}$ for all $1 \leq i \leq n$. Prove the following statements.

   (1) (15%) (Gershgorin circle theorem) Every eigenvalue of A lies in D_i for some $1 \leq i \leq n$.

   (2) (5%) If $|a_{ii}| > r_i$ for all $1 \leq i \leq n$, then A is invertible.