國立高雄大學 108 學年度研究所碩士班招生考試試題

系所：應用數學系

科目：線性代數　　身份別：一般生應用數學組、在職生應用數學組　　是否使用計算機：否

考試時間：100 分鐘　　本科原始成績：100 分

1. (6%) Consider a 3×3 matrix $A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ and a vector $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ c \end{pmatrix}$. Find the value of c that gives a consistent linear system of $A\vec{x} = \vec{b}$.

2. (6%) Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear transformation defined by $T(p(x)) = p(x) + (x+1)\frac{dp(x)}{dx}$, and $\beta = \{1, x, x^2\}$ be a basis for $P_2(\mathbb{R})$. Find the matrix representation of T relative to the given basis β.

3. (6%) Let A be a 3×3 matrix with $\det(A) = -2$, and B be another 3×3 matrix satisfying $\det(-2A^{-1}B^2) = 36$. Find the determinant of B.

4. (6%) Find the volume of the parallelogram formed by the three vectors $\vec{a} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ in $\mathbb{R}^3$.

5. (6%) Consider a 2×2 matrix $C = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ and assume that $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of a 2×2 matrix A. If B is similar to A with $B = C^{-1}AC$, find an eigenvector of B.

6. (6%) Find the projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ on the hyperplane $x_1 - x_2 - x_3 - x_4 = 0$ in $\mathbb{R}^4$.

7. (6%) Let $A = \begin{pmatrix} -3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$, calculate $(A+2I)^4(A+3I)^3$.

8. (10%) Let $\vec{y}(t)$ be a vector valued function satisfying $\frac{d}{dt}\vec{y}(t) = A\vec{y}(t)$ where $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ and $\vec{y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find the general solution of $\vec{y}(t)$.

9. Prove or disprove the following statements:

   (a) (6%) If W_1 and W_2 are two subspaces of $\mathbb{R}^n$, then $W_1 \cup W_2$ is a subspace of $\mathbb{R}^n$.

   (b) (6%) $\mathbb{R}^n$ is a trivial subspace of itself. If $\vec{v}$ is a nonzero vector in $\mathbb{R}^n$, then the subset $\mathbb{R}^n \setminus \vec{v}$ (means that $\mathbb{R}^n$ removes the vector $\vec{v}$) is still a subspace of of $\mathbb{R}^n$.

   (c) (8%) If A is a 2×2 matrix satisfying $A^2 = 2A - I$, then A is diagonalisable.

   (d) (8%) Let $\vec{v} \in \mathbb{R}^m$ and $\vec{w} \in \mathbb{R}^k$ are two nonzero column vectors, and $A = \vec{v}\vec{w}^T$, and $B = \vec{w}\vec{v}^T$. If $m > k$, then $nullity(A) < nullity(B)$.

   (e) (10%) Assume that $A \in \mathbb{C}^{n\times n}$. If A is Hermitian, then all eigenvalues of A are real.

   (f) (10%) If A is a diagonalisable real matrix satisfying $A^3 = A$, then $rank(A) = trace(A^2)$.