※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

In the following, $F^{m\times n}$ denotes the class of all $m\times n$ matrices with entries in the field $F$, where $F=\mathbb{R}$ or $\mathbb{C}$. Vectors in $F^n$ will be regarded as column vectors. We say a matrix $A\in\mathbb{R}^{n\times n}$ is symmetry if $A^T=A$. a matrix $A\in\mathbb{C}^{n\times n}$ is skew-Hermitian if $A^*=-A$. A matrix $A\in\mathbb{R}^{n\times n}$ is positive definite if $x^TAx>0$ for all nonzero $x\in\mathbb{R}^n$.

(1) Let $A\in\mathbb{R}^{2\times 2}$ and write $U=\{X\in\mathbb{R}^{2\times 2}:AX=XA\}$, show that $\dim U\geq 2$. (20 points)

(2) Let $A\in\mathbb{R}^{n\times n}$ be a symmetric matrix and there exists an invertible matrix $R\in\mathbb{R}^{n\times n}$ such that $A=R^TR$, show that $A$ is a positive definite matrix. (20 points)

(3) Let $A=\begin{bmatrix} a & b & c\\ p & q & r\\ u & v & w\end{bmatrix}$ and assume that $\det A=3$.

a. Compute $\det(-A^4)$ (10 points)

b. Compute $\det(2B^{-1})$, where $B=\begin{bmatrix} 2p & -a+u & 3u\\ 2q & -b+v & 3v\\ 2r & -c+w & 3w\end{bmatrix}$. (10 points)

(4) Let $A\in\mathbb{C}^{n\times n}$ be a skew-Hermitian matrix.

a. Show that all eigenvalues of $A$ are purely imaginary. (10 points)

b. If $(\lambda_1,y_1)$ and $(\lambda_2,y_2)$ are two eigenpairs of $A$ with $\lambda_1\neq\lambda_2$, show that $\langle y_1,y_2\rangle=0$. (10 points)

(5) Assume that $A\in\mathbb{R}^{m\times n}$

a. Let $x\in\mathbb{R}^n$, show that if $(A^TA)x=0$, then $Ax=0$. (10 points)

b. Show that Rank $(A^TA)$ =Rank $A$. (10 points)