考試科目	線性代數	所別	應用數學 8111.8116	考試時間	3月6日(六)第二節

* Show all your work.

1. (15 %) Let V be a finite-dimensional vector space over a field F, and let $\beta = \{x_1, x_2, \cdots, x_n\}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from F. Define

$$x'_j = \sum_{i=1}^{n} Q_{ij} x_i, \quad \text{for } 1 \leq j \leq n,$$

and set $\beta' = \{x'_1, x'_2, \cdots, x'_n\}$. Prove that β' is a basis for V and hence that Q is the change of coordinate matrix changing β'-coordinates into β-coordinates.

2. (15 %) Let A be an $m \times n$ matrix with rank m and B be an $n \times p$ matrix with rank n. Determine the rank of AB. Justify your answer.

3. (15 %) Let $A, B \in M_{m\times n}(F)$ be such that $AB = -BA$. Prove that if n is odd and F is not a field of characteristic two, then A or B is not invertible.

4. (20 %) Two linear operators T and U on a finite-dimensional vector space V are called *simultaneously diagonalizable* if there exists an ordered basis β for V such that $[T]_\beta$ and $[U]_\beta$ are diagonal matrices. Prove that if T and U are diagonalizable linear operators on a finite-dimensional vector space V such that $UT = TU$, then T and U are simultaneously diagonalizable.

5. (15 %) Let A be an $n \times n$ matrix with complex entries. Prove that $AA^* = I$ if and only if the rows of A form an orthonormal basis for $\mathbb{C}^n$.

6. (20 %) For the matrix $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & -1 & 3 \end{pmatrix}$, find a basis for each generalized eigenspace of L_A consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of A.

備註	試題隨卷繳交