1. An elastic membrane in the $x_1 x_2$ plane with boundary circle $x_1^2 + x_2^2 = 1$ is stretched so that a point (x_1, x_2) goes over into the point (y_1, y_2) given by

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Ax = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a) Find principal directions (eigenvectors) and their stretch factors (eigenvalues). (5 points)

(b) Plot the un-stretched circle in the $x_1 x_2$ plane and the stretched ellipse in the $y_1 y_2$ plane. (10 points)

(c) $A1$ is similar to A; i.e. $A1 = P^{-1}AP$. Assume $P = \begin{bmatrix} \cos(\frac{\pi}{6}) & \sin(-\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{bmatrix}$ and calculate eigenvalues and eigenvectors of $A1$. (5 points)

(d) Assume $A1 = P^{-1}AP$, and x is eigenvector of A. Prove that $A1$ has the same eigenvalues with A. Eigenvector of $A1$ is $P^{-1}x$ after similarity transformation. (10 points)

2. (a) Based on wave equation in Cartesian coordinates $u_{tt} = c^2 (u_{xx} + u_{yy})$, derive the wave equation in Polar coordinates. (10 points)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

(b) Show how to solve the wave equation in Polar coordinates using the method of separation of variables. Define your notation and derive the following general solution. (10 points)

$$u(r,t) = \sum_{m=1}^{\infty} (A_m \cos\lambda_m t + B_m \sin\lambda_m t) J_0 \left(\frac{\alpha_m}{R} r\right)$$

3. A mass-spring system with two springs in parallel with the mass m = 5 [kg]. The two spring constants are k_1 = 30 and k_2 = 50 [N/m], respectively, and the damping constant is c = 40 [N·s/m].

(a)(10 points)What will its motion, $y(t)$, be if we pull the mass down from rest by 20 cm and let it start with zero initial velocity? Define $y(t)$, form an ODE of $y(t)$, and solve it with initial condition.

(b) Modeling the problem (a) with a linear system of first order ODEs and determine the type and stability of its critical point. (10 points)

(c) Now, the mass spring system is acted upon by a sinusoidal force with amplitude of 40 [N] and a period of π. The force starts from $t = 0$ and ends at $t = \pi$. Use the Laplace transform to solve the output oscillation with initial condition $y(0)=2$, $y'(0) = -3$. (15 points)

4. Solve $y' + y \sin x = e^{\cos x}$, $y(0) = -3$ (5 points)

5. Consider an arbitrary function $y(r)$ that is defined $0 \le r \le 1$ and that obeys $f(1) = 0$. The Dirac delta function has the following property:

$$f(r) = \int_0^1 f(r')\delta(r - r')dr'$$

Using the Sturm-Liouville theory, determine a representation of $\delta(r - r')$ that uses Bessel function J_1 as basis functions. (10 points)

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