國立暨南國際大學 100 學年度碩士班研究生入學考試試題　第 1 節

科目：線性代數　適用：電機所系統組

編號：461

考生注意：
1.依次序作答，只要標明題號，不必抄題。
2.答案必須寫在答案卷上，否則不予計分。
3.限用藍、黑色筆作答；試題須隨卷繳回。

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1. State whether the following statements are true or false. (20%)

   (a) Let V be a vector space. Any set of vectors in V that contains the zero vector is linearly dependent. (5%)

   (b) It takes at least three vectors to span R^3. (5%)

   (c) Every vector space has a unique set of vectors that spans it. (5%)

   (d) Every subset of a linearly dependent set is linearly dependent. (5%)

2. Consider a symmetric matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, which is orthogonally diagonalizable such that $D = P^T AP$ is a diagonal matrix. (15%)

   (a) Find the matrix D. (5%)

   (b) Find the matrix P. (5%)

   (c) Find the matrix A^5. (5%)

3. Consider the vector space R^2 with the inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 + 4y_1y_2$. (15%)

   (a) Determine the norm of the vector $(3, -1)$ in this space. (5%)

   (b) Show that the vectors $(2, 1)$ and $(-8, 4)$ are orthogonal in this space. (5%)

   (c) Determine the distance between the points $(3, -1)$ and $(2, 5)$ in this space. (5%)

4. Let λ_1, λ_2 and λ_3 be all the eigenvalues of the matrix $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 5 & 0 \\ 0 & -1 & 2 \end{bmatrix}$. Find the sum $\lambda_1^3 + \lambda_2^3 + \lambda_3^3$. (10%)

5. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix}$. (10%)

   (a) Is A diagonalizable? Is A a singular matrix? (5%)

   (b) Is B diagonalizable? Is B a singular matrix? (5%)

   (Hint: Please explain your answers.)

6. $A = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$ is the transition probability matrix of a Markov chain. $\mathbf{x}_n$ is the state probability vector and $\mathbf{x}_n = A\mathbf{x}_{n-1}$. Suppose the initial state probability vector $\mathbf{x}_0 = [0.2 \quad 0.7 \quad 0.1]^T$. Find $\lim_{n\to\infty} \mathbf{x}_n$. (20%)

7. If $\hat{A}$ is similar to A, $\hat{A}$ and A have the same eigenvalues. Show that if $\mathbf{x}$ is an eigenvector of A, $\mathbf{y} = P^{-1}\mathbf{x}$ is also an eigenvector of $\hat{A}$ for the same corresponding eigenvalue, where $\hat{A} = P^{-1}AP$. (10%)