國立中山大學100學年度碩士班招生考試試題

科目：工程數學甲【通訊所碩士班乙組】

Problem 1 Multiple Choice (30%)

Instructions:

- There are 10 questions, each of which is associated with 4 possible responses.
- For each of questions, select **ONE most appropriate** response.
- For each response you provide, **you will be awarded 3 marks if the response is correct and -3 marks if the response is incorrect (答錯一題倒扣三分).**
- You get 0 mark if no response is provided.

(1.1) What is the solution of the ODE $\frac{dy}{dx} = 2xy^2$ for which y(0)=1?

(A) $y(x) = (1+x)^{-1}$. (B) $y(x) = (1-x^2)^{-1}$. (C) $y(x) = e^{-x}$. (D) $y(x) = 1+\sqrt[3]{x^4}$.

(1.2) What is the amplitude of the sinusoidal solution of $\frac{dx}{dt} + 2x = 5\sin(3t)$?

(A) $\frac{2}{5}$ (B) 1 (C) $\sqrt{7}$ (D) $\frac{5}{\sqrt{13}}$

(1.3) Let $\boldsymbol{L}[\cdot]$ denotes the Laplace transform.

(A) The Laplace transform is a linear operation.

(B) If $\boldsymbol{L}[f(t)] = F(s)$, then $\boldsymbol{L}[f(t-1)] = F(s-1)$.

(C) $\boldsymbol{L}[(t-2)] = e^{-s}/s$.

(D) None of the above statements is FALSE.

(1.4) What is the general solution to the ODE $t\frac{dx}{dt} = 4t - 3x$?

(A) $x(t) =$ constant. (B) $x(t) = -ct^3$. (C) $x(t) = ct^{-3} + t$. (D) $x(t) = -ce^{3t} + 4t$.

(1.5) Let $\boldsymbol{L}[\cdot]$ denotes the Laplace transform.

(A) $\boldsymbol{L}[t^{-0.5}] = \sqrt{\frac{\pi}{s}}$.

(B) If $\boldsymbol{L}[f(t)] = F(s)$, then $\boldsymbol{L}\left[f\left(\frac{t}{a}\right)\right] = F(as)$.

(C) If $\boldsymbol{L}[f(t)] = F(s)$, then $\boldsymbol{L}\left[\frac{df}{dt}(t)\right] = sF(s)$.

(D) All of the above statements are TRUE.

(1.6) Define the *del operator* $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$, and consider $\varphi(x,y,z) = (\cos z^3)e^{\sqrt{x+y}}$.

(A) $\nabla \cdot \nabla\varphi = \varphi$. (B) $\nabla \times \nabla\varphi = 0$. (C) $\mathbf{R} \cdot \nabla\varphi = 0$, where $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(D) None of the above statements is TRUE.

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(1.7) Consider the del operator defined in (1.6).

(A) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, where **F** is a vector field with continuous first and second derivatives.

(B) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$, where **F** and **G** are two smooth vector fields.

(C) $\nabla \cdot (\varphi \mathbf{F}) = \nabla\varphi \cdot \mathbf{F} + \varphi(\nabla \cdot \mathbf{F})$, where φ and **F** are smooth scalar and vector fields, respectively.

(D) All of the above statements are TRUE.

(1.8) Consider the ODE $\ddot{x} + x^2\dot{x} + x(x^2 - 1) = 0$.

(A) This is a linear ODE.

(B) This is a time-varying ODE.

(C) This ODE has three equilibria.

(D) The equilibria of this ODE are ± 1 and ± 2.

(1.9) Consider the heat equation

$$\frac{\partial u}{\partial t}(x,t) = 4\frac{\partial^2 u}{\partial x^2}(x,t) \qquad \forall 0 < x < 1, t > 0 \qquad (1.9.1)$$

$$u(0,t) = u(1,t) = 0 \qquad \forall t > 0 \qquad (1.9.2)$$

$$u(x,0) = f(x) \qquad \forall 0 < x < 1 \qquad (1.9.3)$$

(A) Without considering the boundary condition (1.9.3), a general solution to the heat equation is $u(x,t) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)\, e^{-4n^2\pi^2 t}$.

(B) Suppose $f(x) = 7\sin(3\pi x)$. Then $u(x,t) = 7\sin(3\pi x)\, e^{-4n^2\pi^2 t}$.

(C) Both of the above statements are TRUE.

(D) None of the above statement is TRUE.

(1.10) Consider the wave equation

$$\frac{\partial^2 w}{\partial t^2}(x,t) = 3\frac{\partial^2 w}{\partial x^2}(x,t) \qquad \forall 0 < x < 1, t > 0 \qquad (1.10.1)$$

$$w(0,t) = w(1,t) = 0 \qquad \forall t > 0 \qquad (1.10.2)$$

$$w(x,0) = f(x) \qquad \forall 0 < x < 1 \qquad (1.10.3)$$

$$\frac{\partial w}{\partial t}(x,0) = 0 \qquad \forall 0 < x < 1 \qquad (1.10.4)$$

(A) Without considering the boundary condition (1.10.3), a general solution to the wave equation is $w(x,t) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)\cos(n\pi t)$.

(B) Suppose $f(x) = 17\sin(9\pi x)$. Then $w(x,t) = 17\sin(9\pi x)\,(\sin(27\pi t) + 1)$.

(C) Both of the above statements are TRUE.

(D) None of the above statement is TRUE.

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Problem 2 (15%)

Consider the vector function $\mathbf{F}(x,y)=(y^3-12y)\mathbf{i}+(15x-x^3)\mathbf{j}$.

(2.1) (10%) Find the simple closed curve C for which the integral

$$\oint_C \mathbf{F}\cdot d\mathbf{r}$$

(with positive orientation) will have the largest positive value. (Hint: Use Green's Theorem)

(2.2) (5%) Compute this largest positive value.

Problem 3 (13%)

This problem has two sub-problems. **Please give your answers in details.**

(a) Let $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\in\mathbb{R}^{2\times 2}$ and define a scalar-valued function $f_A(\mathbf{x},\mathbf{y}):=\mathbf{x}^T A\mathbf{y}$ for any $\mathbf{x},\mathbf{y}\in\mathbb{R}^2$. If $f_A(\mathbf{x},\mathbf{y})$ is an inner product on $\mathbb{R}^2$, then what conclusions can be made on all the entries a_{ij}? (6%)

(b) Consider the inner product space $\left(\mathbb{R}^{2\times 2},\langle\cdot,\cdot\rangle\right)$ with $\langle A,B\rangle := trace(AB^T)$ defined for matrices A and B in $\mathbb{R}^{2\times 2}$. Let S be the subspace of $\left(\mathbb{R}^{2\times 2},\langle\cdot,\cdot\rangle\right)$ defined as $S:=\left\{A\in\mathbb{R}^{2\times 2}\,\middle|\, A=A^T\right\}$. Let E denote the standard basis for S and F denote an orthonormal basis for S that is derived from basis E. What are bases E and F? (7%)

Problem 4 (12%)

Given real numbers a_i, b_i, c_i for $i=1,2$, let L be a transformation from V to W, with $V=\mathbb{R}^2=W$, defined by

$$L(\mathbf{r}) := \begin{bmatrix} a_1 r_1 + b_1 r_2 + c_1 \\ a_2 r_1 + b_2 r_2 + c_2 \end{bmatrix}, \quad \forall\, \mathbf{r} := \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \in \mathbb{R}^2.$$

Suppose that c_1 and c_2 are of the values so that $L:V\to W$ is linear.

以下小題僅需依序寫下答案即可，不需做任何推導。

(a) Let E be the standard basis for $\mathbb{R}^2$. Find the matrix A representing L with respect to basis E for both V and W. (5%)

(b) Let $F:=\{\mathbf{f}_1,\mathbf{f}_2\}$ be another basis for $\mathbb{R}^2$ and let Q denote the matrix representation of L with respect to basis F for V and basis E for W, respectively. Denote $P:=\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$. Please write the algebraic relationship between Q, P, $\mathbf{f}_1$, and $\mathbf{f}_2$. (7%)

Problem 5 (15%)

Compute the least upper bound of the integral

$$\left|\int_C (e^z-\bar{z})\,dz\right|$$

where z is a complex variable, $\bar{z}$ is its complex conjugate, and C denotes the boundary of a triangle with vertices at the points $i3$, -4 and 0, oriented in counterclockwise direction.

Problem 6 (15%)

Let $F(\omega)$ be the Fourier transform of $f(t)$. Compute $\mathscr{F}(i\cdot t\cdot f(t))$, where $\mathscr{F}$ stands for the Fourier transform and $i=\sqrt{-1}$. Write down you answer in terms of $F(\omega)$, and each calculation step is also required.